SIAM J. APPL. MATH.
Vol. 17, No. 1, January 1969

STOCHASTIC PROGRAMS WITH RECOURSE II: ON THE CONTINUITY OF THE OBJECTIVE*

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1. Introduction. In an earlier paper [2] we introduced a general class of stochastic (linear) programs called *stochastic programs with recourse*; for a complete description of the model the reader is referred to Sections 1 and 2 of [2]. One of the principal results of [2] was that the objective of such a program is convex when considered as a deterministic function of the first-stage decision variables. The main result of this paper is that the objective is also lower semicontinuous. Except for the definition of a stochastic program with recourse and a few elementary facts which are repeated here, this paper is largely independent of [2].

The proof of the main result relies on the following lemma of general interest in the theory of convex functions.

LEMMA. *Suppose $f(x, \xi)$ is a function with range in the extended real numbers which is lower semicontinuous and convex in x on R^n and measurable in ξ with respect to a measure μ. Then*

$$f(x) = \int f(x, \xi)\, d\mu$$

is a lower semicontinuous convex function provided only that $f(x) > -\infty$ for all x.

The definitions of lower semicontinuity, convexity, and integrals for functions with range in the extended real numbers and the proof of the lemma are given in § 2.

Section 3 contains the proof of the following main result.

THEOREM 1. *The objective of the equivalent deterministic program of a stochastic program with recourse is a lower semicontinuous convex function provided it is greater than $-\infty$ everywhere.*

Finally, in § 4 we give an example which illustrates the various concepts involved and shows that the objective function need not be upper semicontinuous.

2. Proof of the lemma. In the statement of the lemma we used the familiar concepts of convexity, lower semicontinuity, and integration in the not so familiar context of functions with range in the extended real numbers. We start therefore by giving precise meaning to these concepts.

DEFINITION 1. Let f be a function with R^n for domain and the extended real numbers $\bar{R}$ for range. The set

$$\operatorname{epi} f = \{(z, x) | z \in R, x \in R^n, z \geqq f(x)\}$$

* Received by the editors January 17, 1968.

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is called the *epigraph* of f. The function f is said to be *convex* if its epigraph is a convex subset of R^{n+1} or equivalently if

$$f(x_\lambda) = f[(1-\lambda)x_0 + \lambda x_1] \leqq (1-\lambda)f(x_0) + \lambda f(x_1)$$

for all λ in $[0, 1]$ and x_0, x_1 in R^n, where the conventions $0 \cdot \infty = 0$ and $(+\infty) + (-\infty) = +\infty$ apply. The function f is said to be *lower semicontinuous* if for every converging sequence $\{x^i\}$ in R^n

$$\liminf f(x^i) \geqq f(\lim x^i)$$

or equivalently if epi f is a closed subset of R^{n+1}.

In [2] we extended the definition of integration to allow for functions with infinite values. For convenience we reproduce the definition here. Let $(\Xi, \mathscr{F})$ be a measure space, where $\mathscr{F}$ denotes a σ-algebra on Ξ, and let μ be a (nonnegative) measure defined on it.

DEFINITION 2. Let ψ be a measurable function from Ξ into $\bar{R}$. The *integral* $\int \psi(\xi)\,d\mu$ is defined to be the sum of the following four terms:

$$A[\psi] = \int_{0 \leqq \psi(\xi) < +\infty} \psi(\xi)\,d\mu,$$

$$B[\psi] = \int_{-\infty < \psi(\xi) < 0} \psi(\xi)\,d\mu,$$

$$C[\psi] = \begin{cases} +\infty & \text{if} \quad \mu\{\xi | \psi(\xi) = +\infty\} > 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$D[\psi] = \begin{cases} -\infty & \text{if} \quad \mu\{\xi | \psi(\xi) = -\infty\} > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where again the convention that $(+\infty) + (-\infty) = +\infty$ applies.

In Appendix C of [2] the three following elementary propositions are established.

PROPOSITION 1. *The integral is order preserving, i.e., if f and g are measurable functions from R^n into $\bar{R}$ and $f(\xi) \leqq g(\xi)$ for all ξ in R^n, then* $\int f(\xi)\,d\mu \leqq \int g(\xi)\,d\mu$.

PROPOSITION 2. *The integral is subadditive; specifically, if f and g are measurable functions from R^n into $\bar{R}$, then*

$$\int [f(\xi) + g(\xi)]\,d\mu \leqq \int f(\xi)\,d\mu + \int g(\xi)\,d\mu.$$

Moreover, equality holds except possibly when the integrals on the right are of opposite sign and both are infinite.

An immediate consequence of the first two propositions and the definition of convex function is the next proposition.

PROPOSITION 3. *Suppose $f(x, \xi)$ is a function from $R^n \times \Xi$ into $\bar{R}$ which is convex in x on R^n and measurable in ξ with respect to μ on Ξ. Then $f(x) = \int f(x, \xi)\, d\mu$ is also convex in x.*

The proof of the lower semicontinuity of $f(x)$ in the lemma will employ a characterization of lower semicontinuous convex functions given in Proposition 5 below. An equivalent result obtained with slightly different emphasis may be found in [1]. We shall say that a subset C of a linear space is *linearly closed* if the intersection of C with every line is closed. Since any point on the relative boundary of a convex subset C of R^n is the endpoint of an open line segment contained in the relative interior of C, Proposition 4 follows.

PROPOSITION 4. *A convex subset C of R^n is closed if and only if it is linearly closed.*

PROPOSITION 5. *A convex function f with domain R^n and range $\bar{R}$ is lower semicontinuous if and only if for every line L in R^n the restriction of f to $L \cap S$ is continuous, where S is the closure of the subset of R^n on which f is less than $+\infty$.*

Proof. The proposition follows from Proposition 4, the characterization of lower semicontinuous functions as functions with closed epigraphs, and the observation that a convex function with domain R^1 is lower semicontinuous if and only if its restriction to the closure of the set where it is less than $+\infty$ is continuous.

Proof of the lemma. The convexity of $f(x)$ has already been established. In view of Proposition 5 and the observation that a convex function on R^1 is always continuous on the interior of the interval where it is less than $+\infty$, it will suffice to show that the restriction of

$$f(x_\lambda) = h(\lambda) = \int h(\lambda, \xi)\, d\mu = \int f(x_\lambda, \xi)\, d\mu$$

to the interval $[0, 1]$ is continuous at $\lambda = 0$, where $x_\lambda = (1 - \lambda)x_0 + \lambda x_1$ and x_0, x_1 are points of the relative boundary and relative interior, respectively, of the convex subset of R^n on which $f(x)$ is less than $+\infty$. Let

$$\Sigma_0 = \{\xi \mid h(\lambda, \xi) = +\infty \text{ for all } \lambda \leqq \lambda^0 \text{ with } \lambda^0 \in (0, 1]\},$$

$$\Sigma_1 = \{\xi \mid h(\lambda, \xi) = +\infty \text{ for all } \lambda \geqq \lambda^1 \text{ with } \lambda^1 \in (0, 1]\}.$$

Since, for each ξ, $h(\lambda, \xi) < +\infty$ on a (possibly empty) interval and $h(\lambda) < +\infty$ on $(0, 1]$, it follows from the properties of the integral that Σ_0 and Σ_1 are sets of measure zero. Since $f(x, \xi)$ is lower semicontinuous in x, we have that for each ξ in $S = \Xi - (\Sigma_0 \cup \Sigma_1)$ the function $h(\lambda, \xi) = f(x_\lambda, \xi)$ is continuous, convex for $0 \leqq \lambda \leqq 1$ and less than $+\infty$ except possibly at $\lambda = 0$. Thus,

$$h(\lambda, \xi) = 2(1 - \lambda)h(\tfrac{1}{2}, \xi) - (1 - 2\lambda)h(1, \xi) + \Delta(\lambda, \xi), \tag{1}$$

where for each ξ in S the function $\Delta(\lambda, \xi)$ is continuous on $[0, 1]$ and finite, non-negative and monotonically decreasing on $(0, \tfrac{1}{2}]$. By the monotone convergence theorem, $\Delta(\lambda) = \int_S \Delta(\lambda,\xi)\, d\mu$ is continuous at $\lambda = 0$. (It is possible that the integral for $\Delta(0)$ diverges, but only if $\Delta(\lambda) \to \infty$ as $\lambda \to 0$.) Now if $f(x)$ is nowhere equal to $-\infty$, then both $h(\frac{1}{2})$ and $h(1)$ are finite and we may integrate both sides of (1) and apply Proposition 2 to obtain

$$h(\lambda) = 2(1 - \lambda)h(\tfrac{1}{2}) - (1 - 2\lambda)h(1) - \Delta(\lambda).$$

Since this establishes the continuity of $f(x_\lambda)$ at $\lambda = 0$, the proof of the lemma is complete.

3. Proof of Theorem 1. In view of the linearity of the term $\bar{c}x$ in the objective

$$z(x) = \bar{c}x + Q(x)$$

of the equivalent deterministic program [2, (2.7)] of a stochastic program with recourse, it will suffice to prove the theorem for the function $Q(x)$. In fact, we establish Theorem 1 by proving the following somewhat stronger version.

THEOREM 1′. *Either $Q(x)$ is a lower semicontinuous convex function greater than $-\infty$ everywhere, or $Q(x)$ equals $-\infty$ throughout the relative interior of the convex set K_2^s on which it is less than $+\infty$.*

Proof. By definition $Q(x)$ is the expectation of $Q(x, \xi)$ with respect to μ, where ξ is the vector of random variables of the stochastic program, μ is the probability measure associated with the variables ξ, and $Q(x, \xi)$ is the optimal value of the second-stage linear program [2, (2.2)] as a function of ξ and the vector x of first-stage decision variables. Proposition 4.3 of [2] shows that for each value of ξ the epigraph of $Q(x, \xi)$ is a convex polyhedron. Hence $Q(x, \xi)$ is a convex lower semicontinuous function in x for each ξ. By Lemma 2.3 of [2], $Q(x, \xi)$ is measurable in ξ with respect to μ. Thus by the lemma, $Q(x)$ is a convex lower semicontinuous function unless $Q(x) = -\infty$ for some value of x. But if any convex function takes on the value $-\infty$, it does so throughout the relative interior of the convex set on which it is less than $+\infty$.

4. $Q(x)$ need not be continuous. It is easy enough to construct examples of convex functions defined on convex subsets of R^n which are lower semicontinuous but are not upper semicontinuous on these sets. However, this does not immediately settle the question whether the convex function $Q(x)$ associated with a stochastic program with recourse may be discontinuous on the set where it is less than $+\infty$. The following example shows not only that $Q(x)$ may be discontinuous but that this may happen even if $Q(x) > -\infty$ for all x and the coefficients of the recourse matrix W of the stochastic program are fixed. Of course, as Theorem 4.5 of [2] shows, in any such example the random variables of the stochastic program must fail to have finite variances. As a by-product this example shows that the set K_2^s on which $Q(x)$ is less than $+\infty$ need not be closed. For the remainder of this paper we make free use of the terminology and notation of [2].

We consider the stochastic program with recourse

$$\inf_x E\{\min_y (y_1)\},$$

$$\xi_1 x_1 - y_1 - y_2 = 0,$$

$$\xi_2 x_2 - y_2 - y_3 = 0,$$

$$x_1, x_2, y_1, y_2, y_3 \geqq 0,$$

where Ξ is a countable set of points $(\xi_1(n), \xi_2(n))$, $n = 1, 2, \cdots$, and

$$\xi_1(n) = 2^{2n}, \qquad \xi_2(n) = 2^{2n} - 2^n,$$

$$\mu(\xi_1(n), \xi_2(n)) = 2^{-n}.$$

Clearly $K_2 = \{x | x \geqq 0\}$. Let $Q'(x_1, \xi)$ denote the restriction of $Q(x, \xi)$ to the line $\{x | x_2 = 1\}$. We have

$$Q'(x_1, \xi(n)) = \begin{cases} +\infty & \text{if } x_1 < 0, \\ 0 & \text{if } 0 \leqq x_1 \leqq 1 - 2^{-n}, \\ 2^{2n}(x_1 - 1 + 2^{-n}) & \text{if } x_1 \geqq 1 - 2^{-n}. \end{cases}$$

Note that $Q'(x_1, \xi(n)) = 2^n$ if $x_1 = 1$. Now

$$Q'(x_1) = \sum_{n=1}^{\infty} Q'(x_1, \xi(n)) \cdot 2^{-n}$$

is a finite sum if $0 \leqq x_1 < 1$; but for $x_1 \geqq 1$ each term is at least 1 and the sum diverges. The region P shown striped in Fig. 1 is the epigraph of $Q'(x_1)$. Since p is

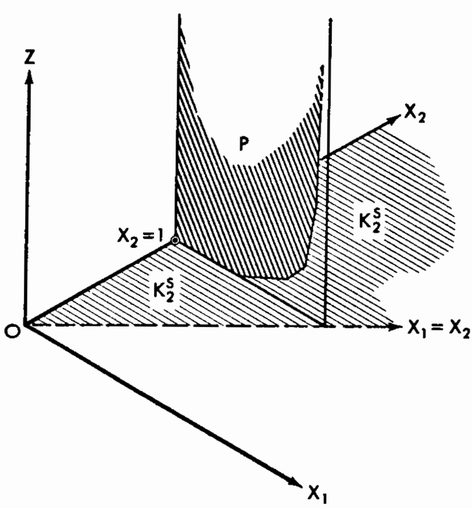

FIG. 1

0 in the stochastic program, $Q(x, \xi)$ and $Q(x)$ are positively homogeneous in x and the epigraph of $Q(x)$ is the closed convex cone with vertex 0 generated by P. The set K_2^s is the projection of the epigraph of $Q(x)$ into the x-plane; specifically,

$$K_2^s = \{x | x_2 > 0, 0 \leqq x_1 < x_2\} \cup \{x | x_1 = x_2 = 0\}.$$

It is easy to see that there are arbitrarily large values of $Q(x)$ in K_2^s in every neighborhood of the point $x = (0, 0)$, specifically, near the boundary $\{x | x_1 = x_2\}$. However $Q(0, 0) = 0$; thus $Q(x)$ is not upper semicontinuous at the point $x = (0, 0)$.

REFERENCES

[1] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, to appear.
[2] D. W. Walkup and R. J.-B. Wets, *Stochastic programs with recourse*, this Journal, 15 (1967), pp. 1299–1314.