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Sharp Bounds on the Value of Perfect Information

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We present sharp bounds on the value of perfect information for static and dynamic simple recourse stochastic programming problems. The bounds are sharper than the available bounds based on Jensen's inequality. The new bounds use some recent extensions of Jensen's upper bound and the Edmundson-Madansky lower bound on the expectation of a concave function of several random variables. Bounds are obtained for nonlinear return functions and linear and strictly increasing concave utility functions for static and dynamic problems. When the random variables are jointly dependent, the Edmundson-Madansky type bound must be replaced by a less sharp "feasible point" bound. Bounds that use constructs from mean-variance analysis are also presented. With independent random variables the calculation of the bounds generally involves several simple univariate numerical integrations and the solution of several similar nonlinear programs. These bounds may be made as sharp as desired with increasing computational effort. The bounds are illustrated on a well-known problem in the literature and on a portfolio selection problem.

LET $f(r, x)$ be the net return in dollars received by the decision maker when he makes decision x and a random variable is observed to be r. Assume that the decision must be made before knowledge of the actual value of the random variable is received. Suppose that the decision maker is an expected utility maximizer and that his utility function over wealth is u. An optimal decision may be found by solving $\max \{E_r u[f(r, x)] \mid x \epsilon K\}$, where it is assumed that r is a random variable with domain $R \subset \mathcal{R}^S$ (Euclidean S-space, $\mathcal{R}$ denotes $\mathcal{R}^1$), $K \subset \mathcal{R}^T$ and E_r represents mathematical expectation with respect to r. Bars above random variables denote their means.

If the decision maker has access to a market survey, clairvoyant, or other way of knowing in advance the actual value of the random variable that will occur, then he may in effect choose an optimal decision conditional on this information. His maximum expected utility is then $E_r \max \{u[f(r, x)] \mid x \epsilon K\}$. The expected value of perfect information, *EVPI*, is the value that equates the maximum attainable expected utility with and without the perfect information. Hence *EVPI* is a solution of

$$\max \{E_r u[f(r, x)] \mid x \epsilon K\} = E_r \max \{u[f(r, x) - EVPI] \mid x \epsilon K\}$$

and represents the maximum amount an expected utility maximizer would be willing to pay for perfect information concerning the random variable.

If we make the simplifying assumption that u is linear, i.e., $u(w) = aw + b$, $a > 0$, then

$$EVPI = E_r \max \{f(r, x) \mid x \epsilon K\} - \max \{E_r f(r, x) \mid x \epsilon K\}. \quad (1)$$

Avriel and Williams [1] have noted that the solution of (1) requires the solution of the stochastic programs

$$Z_n \equiv \max \{E_r f(r, x) \mid x \epsilon K\} \quad (2)$$

and

$$Z_p \equiv E_r \max \{f(r, x) \mid x \epsilon K\}. \quad (3)$$

Since both of these problems are very difficult to solve, Avriel and Williams suggest that one should obtain bounds on *EVPI*. They showed that $EVPI \geqq 0$ as long as all the expected values and maxima in (2) and (3) exist and are finite. We assume throughout the paper that all indicated expected values and maxima exist and are finite. To obtain an upper bound on *EVPI*, they provided an upper bound on Z_p and a lower bound on Z_n. The upper bound is a consequence of

Jensen's Inequality. If $h(y) : Y \rightarrow \mathcal{R}$ is a concave function defined on the convex set $Y \subset \mathcal{R}^n$, then $h(\bar{y}) \geqq E_y h(y)$, and the following well-known results summarized in

LEMMA 1. *Let* $v(x) = max_{y \epsilon D}\, w(x, y)$ *where* $w : C \times D \rightarrow \mathcal{R}$, $v : C \rightarrow \mathcal{R}$, $C \subset \mathcal{R}^n$, $D \subset \mathcal{R}^k$. *Suppose that for each* $x \epsilon C$ *the maximum is obtained for some* $y_0(x) \epsilon D$.

(a) *If* w *is a convex function of* x *for all fixed* $y \epsilon D$ *and* C *is a convex set, then* v *is a convex function of* x.

(b) *If* w *is a concave function of* (x, y) *on the convex set* $C \times D$, *then* v *is a concave function of* x.

(c) *If* w *is a continuous function of* (x, y), *then* v *is a continuous function of* x.

Thus if $\phi(r) \equiv \max \{f(r, x) \mid x \epsilon K\}$ and it is assumed that f is concave in (r, x) on the convex set $R \times K$, it follows from Lemma 1(b) and Jensen's inequality that

$$\begin{aligned} EVPI &\leqq \max\{f(\bar{r}, x) \mid x \epsilon K\} - \max\{E_r f(r, x) \mid x \epsilon K\} \\ &\leqq f(\bar{r}, \hat{x}) - E_r f(r, \hat{x}) \end{aligned} \tag{4}$$

where $\hat{x}$ is an optimal solution to $\max\{f(\bar{r}, x) \mid x \epsilon K\}$. Hence the bound (4) may be computed by solving one nonlinear program and completing one numerical integration.

In general, the bound (4) is not extremely sharp (some numerical results appear later) because it relies on a result that holds for all concave functions and uses a feasible point argument. It is of interest to obtain sharper upper bounds for Z_p and sharper lower bounds for Z_n. We now develop such bounds by using some recent extensions of Jensen's inequality and the Edmundson-Madansky lower bound on the expectation of a concave function of several random variables. The bounds yield sharp upper and lower bounds to both Z_p and Z_n ; hence sharp upper and lower bounds for $EVPI$ are then available.

Assume that f is a continuous concave function of (r, x) on the convex set $R \times K$, where $R \equiv [a, b] \subset \mathcal{R}$. The following theorem[1] is proved in Huang, Ziemba and Ben-Tal [5].

THEOREM 1. *Suppose the interval* $[a, b]$ *is subdivided at arbitrary points* $d_0, \cdots, d_n$, *where* $a = d_0 < d_1 < \cdots < d_{n-1} < d_n = b$. *Let* $\bar{\phi} \equiv \int_a^b \phi(t)\, dF(t)$, *where* ϕ *is a continuous concave function of* $t \epsilon [a, b]$ *and* t *has distribution function* F. *Let* J^n *and* M^n *denote the n-fold generalized Jensen and Edmundson-Madansky bounds, respectively.*

(a) *Let* $J^n \equiv \sum_{i=1}^n \alpha_i \phi(\beta_i)$, $M^n \equiv \sum_{i=0}^n \delta_i \phi(d_i)$, $n = 1, 2, \cdots$. *Then*

$$J^n \geqq \bar{\phi} \geqq M^n, \qquad n = 1, 2, \cdots \tag{5}$$

where $\alpha_i \equiv \int_{d_{i-1}}^{d_i} dF(t) > 0$, $\beta_i \equiv (1/\alpha_i) \int_{d_{i-1}}^{d_i} t\, dF(t)$, $i = 1, \cdots, n$, $\delta_i \equiv \alpha_i[(\beta_i - d_{i-1})/(d_i - d_{i-1})] + \alpha_{i+1}[(d_{i+1} - \beta_{i+1})/(d_{i+1} - d_i)]$, $i = 0, \cdots, n$, *and* $\alpha_0 = \alpha_{n+1} = 0$. *Suppose the partition* P_{k+1} *corresponding to* $k + 1$ *is at least as fine as that corresponding to* k, *i.e.*, $P_k \subset P_{k+1}$ *for* $k = 1, \cdots, n - 1$. *Then*

(b) $$J^1 \geqq \cdots \geqq J^n \geqq \bar{\phi} \geqq M^n \geqq \cdots \geqq M^1,$$

and

(c) $$\lim_{n\to\infty} J^n = \bar{\phi} = \lim_{n\to\infty} M^n$$

(assuming that each subinterval becomes arbitrarily small as $n \to \infty$*).*

[1] See [5] for more details and motivation. The J^m bounds are valid on $\mathcal{R}$; however, the M^m bounds are valid on $\mathcal{R}$ only if $\lim_{t\to\pm\infty} \phi(t)/t$ exists. It is convenient to assume that $R = [a, b]$; the extension to infinite range is straightforward. Particular attention is paid in [5] to the d_i chosen to be iterated partial means. The bounds are reversed when ϕ is convex.

THEOREM 2. *Suppose f is a continuous concave function*[2] *of (r, x) on the convex set $R \times K$, where $R \equiv [a, b]$ and K is a compact set. Then*

(a) $$\max_x L_l(f) \geqq Z_n \geqq \max_x U_l(f);$$

(b) $$L_l(\max_x f) \geqq Z_p \geqq U_l(\max_x f);$$

(c) $$\max\,[0, U_l(\max_x f) - \max_x L_l(f)] \leqq EVPI \leqq L_l\,(\max_x f) - \max_x U_l(f);$$

(d) *Assuming $P_l \subset P_{l+1}$, the bounds for $l + 1$ are at least as sharp as those for l, for all l, and*

(e) $$\lim_{l\to\infty}\,[U_l[\max_x f] - \max_x L_l(f)] = EVPI = \lim_{l\to\infty}[L_l[\max_x f] - \max_x U_l(f)]$$

(assuming that each subinterval becomes arbitrarily small as $l \to \infty$), where $L_l(f) \equiv \sum_{i=1}^{l} \alpha_i f(\beta_i, x)$, $U_l(f) \equiv \sum_{i=0}^{l} \delta_i f\,(d_i, x)$, $L_l(\max_x f) \equiv \sum_{i=1}^{l} \alpha_i \max_x f(\beta_i, x)$, $U_l[\max_x f] \equiv \sum_{i=0}^{l} \delta_i \max_x f\,(d_i, x)$, α_i, β_i, and δ_i are as defined in Theorem 1, *and all maxima are taken over the set $K \subset \mathcal{R}^T$.*

Proof. (a), (b), (c), (d) follow directly from Lemma 1 and Theorem 1. For (e) it suffices to show that

$$\lim_{l\to\infty} L_l(\max_x f) = Z_p = \lim_{l\to\infty} U_l(\max_x f), \tag{6}$$

and

$$\lim_{l\to\infty} \max_x L_l(f) = Z_n = \lim_{l\to\infty} \max_x U_l(f). \tag{7}$$

By Lemma 1, $\max_x f$ is a continuous concave function; hence (6) follows immediately from Theorem 1c. For any $x \epsilon K$ we have by Theorem 1c that

$$\lim_{l\to\infty} \sum_{i=1}^{l} \alpha_i f(\beta_i, x) = E_r f(r, x) = \lim_{l\to\infty} \sum_{i=1}^{l} \delta_i f\,(d_i, x). \tag{8}$$

Suppose x^* solves $\max\{E_r f(r, x) \mid x \epsilon K\}$; then by (8) $\lim_{l\to\infty} \sum_{i=1}^{l} \alpha_i f(\beta_i, x^*) = Z_n = \lim_{l\to\infty} \sum_{i=1}^{l} \delta_i f_i\,(d_i, x^*)$. Thus (7) follows as long as x^* solves $\lim \max_x L_l(f)$ and $\lim \max_x U_l(f)$. This follows because $\lim U_l[f(x^*)] = Z_n \geqq \lim \max_x U_l(f)$, and $\lim \max_x L_l(f) \equiv \lim L_l[f(\hat{x})] = E_r f(r, \hat{x}) \leqq \max_x E_r f(r, x) \equiv E_r f(r, x^*) = \lim L_l[f(x^*)]$.

1. THE DYNAMIC CASE WITH LINEAR UTILITY

We consider the problem in which the decision maker takes a sequence of decisions $\{x_n\}$ in periods $n = 1, \cdots, N$. After each decision is taken, a random variable is observed to be r_n. The net return from decisions $\{x_n\}$ and random realizations $\{r_n\}$ is assumed to be $f(r_1, \cdots, r_N, x_1, \cdots, x_N) \equiv f(r^N, x^N)$ using the notation $r^N \equiv (r_1, \cdots, r_N)$ and $x^N \equiv (x_1, \cdots, x_N)$. Initially we assume that the $\{r_n\}$ are independent and that the investor's utility function over f is linear. The no-information and perfect-

[2] Similar results may be obtained when f is a convex function of r for all fixed values of $x \epsilon K$ and is continuous on $R \times K$. The bounds and the role of U_l and L_l are then reversed.

information problems are then $Z_n^N \equiv \max_{x_1 \epsilon K_1} E_{r_1} \max_{x_2 \epsilon K_2} \cdots \max_{x_N \epsilon K_n} E_{r_N} f(r^N, x^N)$, and $Z_p^N \equiv E_{r_1} \cdots E_{r_N} \max_{x_1 \epsilon K_1} \cdots \max_{x_N \epsilon K_n} f(r^N, x^N)$, respectively. Hence $EVPI = Z_p^N - Z_n^N$. For expositional ease we present the results and proofs of Theorems 3 and 4 for the case $N = 2$. These results can easily be generalized to the case of arbitrary finite N (see [4] for specific details).

THEOREM 3. *Suppose f is a continuous concave function of (r^2, x^2), where $r^2 \epsilon [a_1, b_1] \times [a_2, b_2]$, $x^2 \epsilon K^2 = K_1 \times K_2$ and K^2 is a compact set. Then*

$$\text{(a)} \quad L_{n,l} \equiv \max_{x_1} L_{l_1} [\max_{x_2} L_{l_2}(f)] \geqq Z_n^2 \geqq \max_{x_1} U_{l_1} [\max_{x_2} U_{l_2}(f)] \equiv U_{n,l}, \tag{9}$$

$$\text{(b)} \quad L_{p,l} \equiv L_{l_1} [L_{l_2} [\max_{x^2} f]] \geqq Z_p^2 \geqq U_{l_1}[U_{l_2} [\max_{x^2} f]] \equiv U_{p,l},$$

and (c) $$\max [0, U_{p,l} - L_{n,l}] \leqq EVPI \leqq L_{p,l} - U_{n,l}.$$

(d) *If $\hat{l} \geqq l$, then the bounds in (c) corresponding to $\hat{l}$ are at least as sharp as those corresponding to l assuming $P_l \subset P_{\hat{l}}$, and*

$$\text{(e)} \quad \lim_{l \to \infty} [U_{p,l} - L_{n,l}] = EVPI = \lim_{l \to \infty} [L_{p,l} - U_{p,l}]$$

(*assuming that each subinterval becomes arbitrarily small as l_1 and $l_2 \to \infty$*).

Proof. (a) Since f is continuous and concave in $r_2 \epsilon [a_2, b_2]$, applying (5) to $E_{r_2} f$ yields $Z_n^2 \leqq \max_{x_1} E_{r_1} [\max_{x_2} L_{l_2}(f)]$, where $L_{l_2}(f)$ is a continuous concave function of (r_1, x^2). Thus by Lemma 1, $\max_{x_2} L_{l_2}(f)$ is continuous and concave in (r_1, x_1). Applying (5) to $E_{r_1}[\max_{x_2} L_{l_2}(f)]$ yields $Z_n^2 \leqq \max_{x_1} L_{l_1} [\max_{x_2} L_{l_2}(f)]$. An analogous proof verifies the right-hand side of (a).

(b) Lemma 1 indicates that $\max_{x^2} f$ is continuous and concave in (r_1, r_2). Thus applying (5) to $E_{r_1} [\max_x f]$ yields $Z_p^2 \leqq E_{r_1}[L_{l_2} [\max_{x^2} f]]$, where $L_{l_2} [\max_{x^2} f]$ is continuous and concave in r_1. Thus we can also apply (5) to $E_{r_1}[L_{l_2} [\max_{x^2} f]]$ to obtain $Z_p^2 \leqq L_{l_1}[L_{l_2} [\max_{x^2} f]]$. An analogous proof verifies the right side of (b). Now (c), (d), and (e) follow directly from (a), (b), and Theorem 1.

Since

$$\max_{x_1} \sum_{i=1}^{l} \alpha_i^1 \max_{x_2} \sum_{j=1}^{l_2} \alpha_j^2 f(\beta_i^1, \beta_j^2, x_1, x_2) = \max_{x_1, x_2(\beta_i^1)} \sum_{i=1}^{l_1} \alpha_i^1 \sum_{j=1}^{l_2} \alpha_j^2 f(\beta_i^1, \beta_j^2, x_1, x_2(\beta_i^1)) \tag{10}$$

and $$\max_{x_1} \sum_{i=0}^{l_1} \delta_i^1 \max_{x_2} \sum_{j=0}^{l_2} \delta_j^2 f(d_i^1, d_j^2, x_1, x_2) = \max_{x_1, x_2(d_i^1)} \sum_{i=0}^{l_1} \delta_i^1 \sum_{j=0}^{l_2} \delta_j^2 f(d_i^1, d_j^2, x_1, x_2(d_i^1)), \tag{11}$$

each bound for Z_n^2 may be computed by solving a single finite-dimension nonlinear program. Under the assumptions of Theorem 3, as $l_1, l_2 \to \infty$ it is clear that

THEOREM 4.

$$\max_{x_1} E_{r_1} \max_{x_2} E_{r_2} f(x^2, r^2) = \max_{x_1, x_2(r_1)} E_{r^2} f(x_1, x_2(r_1), r^2).$$

Remark 1. Theorem 3 indicates that both of the bounds in (9) may be computed via the large nonlinear programs (10) and (11). This is a generalization of the argument used to develop a large linear program that is equivalent to a two-stage stochastic program when the random variables have discrete distributions (see, e.g., Dantzig and Madansky [2]). Whether or not the solution of such a large nonlinear program is preferable to a solution via dynamic programming is largely dependent on the cardinality of x^N, in addition to the influence of the K_n, f and the grid sizes.

Remark 2. Rockafellar and Wets [10] have shown that $\max_{x(r)} E_r f(r, x) = E_r \max_x f(r, x)$ under very general conditions that are much weaker than those assumed here. Their result can be used to provide an alternative proof of Theorem 4. The proof using the bounding approach has the advantage of directly indicating the existence of equivalent finite-dimension nonlinear programs to the upper and lower bounds.

When the $\{r_n\}$ are dependent, it is not possible to apply directly the Edmundson-Madansky type bound because although such bounds exist (see Madansky [6]), there is no apparent simple way to calculate them. However, bounds similar to the Avriel-Williams bound (4) may be developed in this case. Ziemba and Butterworth [12] have presented such bounds for the two-period case; the following is a generalization to N-periods. We consider then the no-information and perfect-information problems

$$Z_n^N \equiv \max_{x_1 \epsilon K_1} E_{r_1} \cdots \max_{x_N \epsilon K_N} E_{r_N | r^{N-1}} f(r^N, x^N), \tag{12}$$

and $Z_p^N \equiv E_{r^N} \max_{x^N \epsilon K^N} f(r^N, x^N)$, respectively. Now $EVPI = Z_p^N - Z_n^N$.

THEOREM 5. *Suppose f is a concave function of $(r^N, x^N) \epsilon R^N \times K^N$, where $R^N \equiv R_1 \times \cdots \times R_N$, each R_n is a convex subset of $\mathbb{R}^{S_n}$, $K^N \equiv K_1 \times \cdots \times K_N$, and each K_n is a convex subset of $\mathbb{R}^{T_n}$. Then*

(a) $$0 \leqq EVPI \leqq f(\bar{r}^N, \hat{x}^N) - E_{r^N} f(r^N, \hat{x}^N), \tag{13}$$

where $\hat{x}^N$ solves $\max_{x^N \epsilon K^n} f(\bar{r}^N, x^N)$.

(b) *Suppose, in addition, that each $(\bar{r}_n \mid r^{n-1})$ is a convex (concave) function of r^{n-1} and f is a non-increasing (non-decreasing) function of r^n. Then*

$$0 \leqq EVPI \leqq f(\bar{r}_1, \cdots, (\bar{r}_N \mid \bar{r}^{N-1}), \tilde{x}^N) - E_{r^N} f(r^N, \tilde{x}^N), \tag{14}$$

where $\tilde{x}^N$ solves $\max_{x^N \epsilon K^N} f(\bar{r}_1, \cdots, (\bar{r}_N \mid \bar{r}^{N-1}), x^N)$.

Proof. (a) Under the assumptions all maxima and expectations exist; thus it is easy to show that $EVPI \geqq 0$. Since $\max \{f(r^N, x^N) \mid x^N \epsilon K^N\}$ is concave in r^N, it follows by Jensen's inequality that $EVPI \leqq$

$\{\max f(\bar{r}^N, x^N) \mid x^N \epsilon K^N\} - Z_n^N$ and $EVPI \leqq f(\bar{r}^N, \hat{x}^N) - E_{r^N} f(r^N, \hat{x}^N)$, since $\hat{x}^N$ is feasible for (12).

(b) The proof uses the following lemma, whose proof is found in Mangasarian [8].

Lemma 2. *Let* $v(x) \equiv u[q_1(x), \cdots, q_I(x)]$, *where* $v : C \to R$, $u : \mathfrak{R}^I \to \mathfrak{R}$, $g_i : C \to \mathfrak{R}$, $i = 1, \cdots, I$, *and* $C \subset \mathfrak{R}^n$ *is a convex set. Suppose u is concave and non-increasing (non-decreasing) and each* q_i *is convex (concave). Then v is concave.*

Using the conditional form of Jensen's inequality yields $EVPI \leqq E_{r^{N-1}} \max_{x^N} f[r^{N-1}, (\bar{r}_N \mid r^{N-1}), x^N] - Z_n^N$. By Lemmas 1 and 2 it follows that $\phi(r^{N-1}) \equiv \max \{f[r^{N-1}, (\bar{r}_N \mid r^{N-1}), x^N] \mid x^N \epsilon K^N\}$ is a concave function. Hence by Jensen's inequality $EVPI \leqq E_{r^{N-2}} \max_{x^N} f[r^{N-2}, (\bar{r}_{N-1} \mid r^{N-2}), (\bar{r}_N \mid \bar{r}_{N-1}, r^{N-2}), x^N] - Z_n^N$. Continuing inductively yields $EVPI \leqq \max \{f\{\bar{r}_1, \cdots, \bar{r}_N(\bar{r}^{N-1}), x^N\} \mid x^N \epsilon K^N\} - Z_n^N \leqq f\{\bar{r}_1, \cdots, \bar{r}_N(\bar{r}^{N-1}), \tilde{x}^N\} - E_{r^N} f(r^N, \tilde{x}^N)$ since $\tilde{x}^N$ is feasible for (12).

Remark 3. To apply this type of bound it is necessary to assume that f is concave in r^N. A similar lower bound does not exist in the convex case.

In general, $\bar{r}_n \neq \bar{r}_n(\bar{r}^{n-1})$. Equality is obtained if each $\bar{r}_n(r^{n-1})$ is linear, such as when r^N has a multivariate normal distribution; the bounds in (a) and (b) are then equivalent. If the assumptions of (b) are satisfied, the tightest upper bound will be the minimum of (13) and (14) since neither dominates for general concave f. However, if $\hat{x}^N = \tilde{x}^N$, then (13) (respectively (14)) is the sharper bound when each $(\bar{r}_n \mid r^{n-1})$ is concave (convex) and f is non-decreasing (non-increasing).

To calculate this bound it is necessary to solve one nonlinear program and to compute one numerical integration.

2. THE STATIC CASE WITH SEVERAL RANDOM VARIABLES AND LINEAR UTILITY

In the static case with several random variables the no-information and perfect-information problems may be written as

$$Z_n^Q \equiv \max_{x \epsilon K} E_{r^Q} f(r^Q, x) \tag{15}$$

and

$$Z_p^Q \equiv E_{r^Q} \max_{x \epsilon K} f(r^Q, x) \tag{16}$$

respectively, where $r^Q \equiv (r_1, \cdots, r_Q)$, the r_q have independent distributions on $R_q \equiv [a_q, b_q]$, $x \epsilon K \subset \mathfrak{R}^T$, and K is a compact set. Suppose that f is continuous and concave on $R \times K$, where $R \equiv R_1 \times \cdots \times R_Q$. Clearly, (15) and (16) are merely special cases of the dynamic problems considered in Section 1. Hence one has

Theorem 6. *Under the assumptions stated above*

(a) $$L_{n,m} \geqq Z_n^Q \geqq U_{n,m};$$

(b) $$L_{p,m} \geqq Z_p^Q \geqq U_{p,m};$$

(c) $$\max\,[0, U_{p,m} - U_{n,m}] \leqq EVPI \leqq L_{p,m} - U_{n,m};$$

(d) *If* $\hat{m} \geqq \tilde{m}$, *then the bounds in* (c) *corresponding to* $\hat{m}$ *are at least as sharp as those corresponding to* $\tilde{m}$ *for all* $\tilde{m}$ *and* $\hat{m}$, *assuming* $P_{\tilde{m}} \subset P_{\hat{m}}$; *and*

(e) $$\lim_{m\to\infty} (U_{p,m} - L_{n,m}) = EVPI = \lim_{m\to\infty} (L_{p,m} - U_{n,m}),$$

(*assuming that each subinterval becomes arbitrarily small as each* $m_q \to \infty$),

where

$$\begin{aligned}
L_{n,m} &\equiv \max_x L_{m_1} \cdots L_{m_Q}(f) \\
&\equiv \max_{x\epsilon K} \textstyle\sum_{i_1=1}^{m^1} \alpha_{i_1}^1 \cdots \sum_{i_Q=1}^{m_Q} \alpha_{i_Q}^Q f(\beta_{i_1}^1, \cdots, \beta_{i_Q}^Q, x), \\
U_{n,m} &\equiv \max_x U_{m_1} \cdots U_{m_Q}(f) \\
&\equiv \max_{x\epsilon K} \textstyle\sum_{i_1=0}^{m_1} \delta_{i_1}^1 \cdots \sum_{i_Q=0}^{m_Q} \delta_{i_Q}^Q f\,(d_{i_1}^1, \cdots, d_{i_Q}^Q, x), \\
L_{p,m} &\equiv L_{m_1} \cdots L_{m_Q}\,(\max_x f) \\
&\equiv \textstyle\sum_{i_1=1}^{m^1} \cdots \sum_{i_m=1}^{m_Q} (\pi_{k=1}^Q \alpha_{i_k}^k) \max_{x\epsilon K} f(\beta_{i_1}^1, \cdots, \beta_{i_Q}^Q, x), \\
U_{p,m} &\equiv U_{m_1} \cdots U_{m_Q}\,(\max_x f) \\
&\equiv \textstyle\sum_{i_1=0}^{m_1} \cdots \sum_{i_Q=0}^{m_Q} (\pi_{k=1}^Q \delta_{i_k}^k) \max_{x\epsilon K} f\,(d_{i_1}^1, \cdots, d_{i_Q}^Q, x),
\end{aligned}$$

and $m \equiv (m_1, \cdots, m_Q)$.

The bounds in Theorem 6 may be illustrated on the following problem taken from Mangasarian [7]. Let $f(x, r) = r_1x_1^2 + r_2x_2^2$, $K \equiv \{(x_1, x_2) \mid 0 \leqq x_1 \leqq 10, 0 \leqq x_2 \leqq 5\}$, where r_1 and r_2 have uniform distributions on $[-\tfrac{1}{2}, \tfrac{1}{2}]$ and $[0, 1]$, respectively. One then has the bounds[3] $0 \leqq EVPI \leqq 31.25$, $5.65 \leqq EVPI \leqq 14.07$, $8.98 \leqq EVPI \leqq 12.89$, $10.84 \leqq EVPI \leqq 12.60$, for $l = 0, 1, 2, 3$, respectively, where $l = 0$ uses the Jensen and Edmundson-Madansky bounds and $l = 1, 2, 3$ uses the generalized Jensen and Edmundson-Madansky bounds. Each $m_l = 2^l$ and the partition at $l + 1$ is obtained by subdividing each subinterval at l by its partial mean.

Remark 4. The bounds in Theorems 3 and 6 may be easily generalized to the case where there are multiple random variables and multiple periods. The complexity of the resulting expression in the general case is, however, extremely unwieldly and is perhaps of limited practical importance. Hence it is omitted.

[3] Since f is convex in r for all feasible x, the convexity of $\phi(r) \equiv \max_{x\epsilon K} f(x, r)$ follows from Lemma 1a. The bounds in Theorem 6 then become $\max\,[0, L_{p,m} - U_{n,m}] \leqq EVPI \leqq U_{p,m} - L_{n,m}$.

3. THE STATIC CASE WITH CONCAVE UTILITY

If the decision maker's utility function u is nonlinear, then the expected value of perfect information is a solution of

$$E_r \max \{u[f(r, x) - EVPI] \mid x \epsilon K\} = \max \{E_r u[f(r, x)] \mid x \epsilon K\}. \quad (17)$$

The solution to (17) is known (see Marschak and Radner [9]) to be unique (assuming all indicated expected values and maxima exist) if u is strictly increasing. Ziemba and Butterworth [12] have developed the following bounds on $EVPI$ using Jensen's inequality and a feasible point argument.

THEOREM 7. *Suppose that u is strictly increasing and concave on $\mathcal{R}$ and f is concave on the convex set $R \times K \subset \mathcal{R}^S \times \mathcal{R}^T$. Then $0 \leqq EVPI \leqq f(\bar{r}, \hat{x}) - u^{-1}\{E_r u[f(r, \hat{x})]\}$, where $\hat{x}$ solves $\max \{f(\bar{r}, x) \mid x \epsilon K\}$ and u^{-1} is the inverse function of u.*

This bound may be illustrated on the following portfolio selection problem. Suppose $u(w) = w^{\frac{1}{2}}$, $f(r, x) = r_1 x_1 + r_2 x_2$, $K \equiv \{(x_1, x_2) \mid x_1 + x_2 = 1, x_1 \geqq 0, x_2 \geqq 0\}$ and r_1 and r_2 have uniform distributions on [1, 3] and [2, 3], respectively. Now $\bar{r}_1 = 2$, $\bar{r}_2 = 2.5$, $\hat{x} = (0, 1)$, $f(\bar{r}, \hat{x}) = 2.5$, $Euf(r, x) \cong 1.5784$, and $u^{-1}(1.5784) \cong 2.4910$. Thus $0 \leqq EVPI \leqq 0.0090$.

Unfortunately, this is the only bound using Jensen or Edmundson-Madansky type inequalities that enables us to obtain a simple explicit expression for $EVPI$.[4] However, it is possible to develop bounds using constructs from mean-variance portfolio analysis.

THEOREM 8. *Suppose that*

(i) *u is strictly increasing and concave on $\mathcal{R}$;*

(ii) *$x^* \equiv x^*(r)$ and $\hat{x}$ solve $\max \{f(r, x) \mid x \epsilon K\}$ and $\max \{E_r u[f(r, x)] \mid x \epsilon K\}$, respectively, where $r \epsilon R \subset \mathcal{R}^S$; and*

(iii) *$f(r, x^*)$ and $f(r, \hat{x})$ have distributions belonging to the same family with two parameters that are independent functions of mean and variance, i.e., if $f(r, x^*) \sim G(y : a_1, b_1)$ and $f(r, \hat{x}) \sim H(z : a_2, b_2)$, then $G(y) = H(z)$ whenever $(y - a_1) b_1^{\frac{1}{2}} = (z - a_2)/b_2^{\frac{1}{2}}$, where the a_i are finite and the b_i are positive and finite. Then*

$$\text{(a)} \quad EVPI \geqq (\text{resp.}, \leqq)\ E_r[f(r, x^*) - f(r, \hat{x})] \text{ iff } \operatorname{var} f(r, x^*) \leqq (\text{resp.}, \geqq) \operatorname{var} f(r, \hat{x}), \quad (18)$$

and

(b) *a risk-averse decision maker will never have a lower value of $EVPI$ than a risk-neutral decision maker iff $\operatorname{var} f(r, x^*) \leqq \operatorname{var} f(r, \hat{x})$.*

[4] Specifically, when one of the generalized bounds is applied to the left side of (17), one has an expression of the form $\sum \dot{a}_i \max \{u[f(\beta_i, x)] \mid x \epsilon K\} \geqq$ or $\leqq \max \{E_r u[f(r, x)] \mid x \epsilon K\}$. Hence inversion of u is not possible except in the unlikely case when the solution to $\max \{u[f(\beta_i, x)] \mid x \epsilon K\}$ is independent of i.

Proof. (a) By definition $f(r, x^*) \equiv \max \{f(r, x) \mid x \epsilon K\}$ and $E_r u[f(r, \hat{x})] \equiv \max \{E_r u[f(r, x)] \mid x \epsilon K\}$. Since strictly increasing u implies $E_r \max u[f(r, x) - EVPI] = E_r u[\max_x f(r, x) - EVPI]$, (17) may be written as

$$E_r u[f(r, x^*) - EVPI] = E_r u[f(r, \hat{x})] = E_r u[f(r, x^*) - \{f(r, x^*) - f(r, \hat{x})\}].$$

Let $g(r) \equiv f(r, x^*) - E_r[f(r, x^*) - f(r, \hat{x})]$. Since $g(r)$ and $f(r, \hat{x})$ have distributions belonging to the same family described by two parameters that are independent functions of mean and variance with equal means, it is well known (see, e.g., Hanoch and Levy [3]) that

$$E_r u[g(r)] \geqq (\text{resp.}, \leqq) \ E_r u[f(r, \hat{x})] \text{ iff } \operatorname{var} g(r) \leqq (\text{resp.}, \geqq) \cdot \operatorname{var} f(r, \hat{x}) \quad (19)$$

for all non-decreasing concave u. Therefore,

$$E_r u[f(r, x^*) - E_r[f(r, x^*) - f(r, \hat{x})]] \geqq (\text{resp.}, \leqq) \ E_r u f(r, \hat{x}) = E_r u[f(r, x^*) - EVPI] \text{ iff } \operatorname{var} g(r) \geqq (\text{resp.}, \leqq) \operatorname{var} f(r, \hat{x}).$$

Thus $EVPI \geqq$ (resp., $\leqq$) $E_r[f(r, x^*) - f(r, \hat{x})]$ iff $\operatorname{var} g(r) \leqq$ (resp., $\geqq$) $\operatorname{var} f(r, \hat{x})$. Hence the result follows because $\operatorname{var} g(r) = \operatorname{var} f(r, x^*)$.

(b) Since $EVPI \geqq E_r[f(r, x^*) - f(r, \hat{x})]$ iff $\operatorname{var} f(r, x^*) \leqq \operatorname{var} f(r, \hat{x})$ and $\max_{x \epsilon K} E_r f(r, x) \geqq E_r f(r, \hat{x})$, it follows that $EVPI \geqq E_r f(r, x^*) - \max_{x \epsilon K} E_r f(r, x) \geqq E_r \max_{x \epsilon K} f(r, x) - \max_{x \epsilon K} E_r f(r, x)$.

Remark 5. The results hold for any distribution that has two parameters that are independent functions of mean and variance such as the normal, or the uniform and two-point equally likely distributions (when $S = 1$). The lognormal is a distribution having two parameters that are not independent functions of mean and variance. The results may be extended slightly to apply to stable variates whose mean exists. See Ziemba [11] for an extension of the result in (19) to stable variates that yields (18), letting dispersion replace variance.

To compute the bound in (a), it is necessary to solve one nonlinear program and one stochastic program and to compute a numerical integration.

4. THE DYNAMIC CASE WITH CONCAVE UTILITY

We consider the problem outlined in Section 1 under the assumption that the decision maker has a strictly increasing concave utility function u over $f(r^N, x^N)$. Then $EVPI$ is defined by

$$E_{r^N} \max_{x^N} u[f(r^N, x^N) - EVPI] = \max_{x_1 \epsilon K_1} E_{r_1} \cdots \max_{x_N \epsilon K_N} E_{r_N \mid r^{N-1}} u[f(r^N, x^N)]. \quad (20)$$

The natural generalization of Theorems 5 and 7 is

THEOREM 9. *Suppose* (a)

(i) *u is strictly increasing and concave on* $\mathcal{R}$ *and*

(ii) *f is concave on the convex set* $(r^N, x^N)\epsilon R^N \times K^N$, $R^N \subset \mathcal{R}^N$, *and* $K^N \equiv K_1 \times \cdots \times K_N \subset \mathcal{R}^{T_1} \times \cdots \times \mathcal{R}^{T_N}$. *Then* $0 \leqq EVPI \leqq f(\bar{r}^N, \hat{x}^N) - u^{-1}[E_{r^N}u\{f(r^N, \hat{x}^N)\}]$, *where* $\hat{x}^N$ solves max $[f(\bar{r}^N, x^N) \mid x^N \epsilon K^N]$.

(b) *Suppose in addition that each* $(\bar{r}_n \mid r^{n-1})$ *is a convex (concave) function of* r^{n-1} *and that f is a non-increasing (non-decreasing) function of* r^n. *Then* $0 \leqq EVPI \leqq f(\bar{r}_1, \cdots, \bar{r}_N(\bar{r}^{N-1}), \tilde{x}_N) - u^{-1}[E_{r^N}uf(r^N, \tilde{x}^N)\}]$, *where* $\tilde{x}^N$ *solves max* $[f(\bar{r}_1, \cdots, (\bar{r}_N \mid \bar{r}^{N-1}), x^N\}]$.

Proof. (a) Under the assumptions, $h(r^N, x^N, EVPI) \equiv u[f(r^N, x^N) - EVPI]$ is a concave function of r^N for given $x^N \epsilon K^N$ and $EVPI$. Hence applying Jensen's inequality to (20) yields

$$\max_{x^N \epsilon K^N} u[f(\bar{r}^N, x^N) - EVPI] \geqq \max_{x_1 \epsilon K_1} E_{r_1} \cdots \max_{x_N \epsilon K_N} E_{r_N \mid r^{N-1}} u[f(r^N, x^N)]. \quad (21)$$

Since u is strictly increasing, an optimal solution to $\max_{x^N \epsilon K^N} [f(\bar{r}^N, x^N) - EVPI]$ is an optimal solution to the LHS of (21). This yields the upper bound since $\hat{x}^N$ is feasible for the RHS of (21). It is easy to show that $EVPI \geqq 0$.

(b) The proof is essentially the same as the proof of Theorem 5b, using N sequential applications of Jensen's inequality to the LHS of (21) and the inversion operation. Remark 1 applies here as well.

It is also possible to generalize Theorem 8 to the dynamic case.

THEOREM 10. *Suppose that*

(i) *u is strictly increasing and concave on* $\mathcal{R}$;

(ii) $x^{N*} \equiv x^{N*}(r^N)$ *and* $\hat{x}^N \equiv (\hat{x}_1, \hat{x}_2(r_1), \cdots, \hat{x}_N(r^{N-1}))$ *solve max* $\{f(r^N, x^N) \mid x^N \epsilon K^N\}$ *and* $\max_{x_1 \epsilon K_1} E_{r_1} \cdots \max_{x_N \epsilon K_N} E_{r_N \mid r^{N-1}} u[f(r^N, x^N)]$, *respectively, where* $r^N \epsilon R^N \subset \mathcal{R}^{S_1} \times \cdots \times \mathcal{R}^{S_N}$; *and*

(iii) $f(r^N, x^{N*})$ *and* $f(r^N, \hat{x}^N)$ *have distributions belonging to the same family with two parameters that are independent functions of mean and variance, i.e., if* $f(r^N, x^{N*}) \sim G(y : a_1, b_1)$ *and* $f(r^N, \hat{x}^N) \sim H(z : a_2, b_2)$, *then* $G(y) = H(z)$ *whenever* $(y - a_1)/b_1^{\frac{1}{2}} = (z - a_2)/b_2^{\frac{1}{2}}$, *where the a_i are finite and the b_i are positive and finite. Then*

(a) $EVPI \geqq$ (resp., $\leqq$) $E_r[f(r^N, x^{N*}) - f(r^N, \hat{x}^N]$ iff var $f(r^N, x^{N*}) \leqq$ (resp., $\geqq$) var $f(r^N, \hat{x}^N)$;

(b) A risk-averse decision maker will never have a lower value of $EVPI$ than a risk-neutral decision maker iff var $f(r^N, x^{N*}) \leqq$ var $f(r^N, \hat{x}^N)$.

Proof. The proof is exactly the same as Theorem 8 if we let $f(r, x^*) \equiv f(r^N, x^{N*}) \equiv \{\max f(r^N, x^N) \mid x^N \epsilon K^N\}$, $E_r uf(r, \hat{x}) \equiv E_{r^N} u[f(r^N, \hat{x}^N)] =$

$\max_{x_1 \epsilon K_1} E_{r_1} \cdots \max_{x_N \epsilon K_N} E_{r_N | r_{N-1}} u[f(r^N, x^N)]$, and $g(r) \equiv g(r^N) \equiv f(r^N, x^{N*}) - E_{r^N}[f(r^N, x^{N*}) - f(r^N, \hat{x}^N)]$.

Remark 6. It is not necessary in Theorem 10 to assume that the r^n are independent nor that the $S_n = 1$. Remark 5 also applies to this result.

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REFERENCES

1. M. Avriel and A. C. Williams, "The Value of Information and Stochastic Programming," *Opns. Res.* **18,** 947–954 (1970).
2. G. B. Dantzig and A. Madansky, "On the Solution of Two-Stage Linear Programs Under Uncertainty," *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability,* **1,** J. Neyman (ed.), pp. 165–176, University of California Press, Berkeley, Calif., 1961.
3. G. Hanoch and H. Levy, "The Efficiency Analysis of Choices Involving Risk," *Rev. Econ. Stud.* **36,** 335–346 (1969).
4. C. C. Huang, I. Vertinsky and W. T. Ziemba, "Sharp Bounds on the Value of Perfect Information," U.B.C. Faculty of Commerce W. P. No. 293, September 1975.
5. C. C. Huang, W. T. Ziemba, and A. Ben-Tal, "Bounds on the Expectation of a Convex Function of a Random Variable: with Applications to Stochastic Programming," *Opns. Res.* (in press).
6. A. Madansky, "Bounds on the Expectation of a Convex Function of a Multivariate Random Variable," *Ann. Math. Statistics* **30,** 743–746 (1959).
7. O. L. Mangasarian, "Nonlinear Programming Problems with Stochastic Objective Functions," *Management Sci.* **12,** 353–359 (1964).
8. O. L. Mangasarian, "Convexity, Pseudo-Convexity and Quasi-Convexity of Composite Functions," *Cahiers du Centre d'Etudes de Recherche Operationnelle,* **12,** 114–122 (1970).
9. J. Marschak and R. Radner, *Economic Theory of Teams,* Yale University Press, New Haven, Conn., 1971.
10. R. T. Rockafellar and R. Wets, "Stochastic Convex Programming: Basic Duality," *Pacific J. Math.* (forthcoming).
11. W. T. Ziemba, "Choosing Investments when the Returns have Stable Distributions," *Mathematical Programming in Theory and Practice,* P. L. Hammer and G. Zoutendijk (eds.), pp. 443–475, North Holland Publishing Co., Inc., Amsterdam, 1974.
12. W. T. Ziemba and J. E. Butterworth, "Bounds on the Value of Information in Uncertain Decision Problems," *Stochastics* **1,** 361–378 (1975).