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MATHEMATICS OF OPERATIONS RESEARCH
Vol. 12, No. 1, February 1987
Printed in U.S.A.

COMPUTING BOUNDS FOR STOCHASTIC PROGRAMMING PROBLEMS BY MEANS OF A GENERALIZED MOMENT PROBLEM*†

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Bounds on the expected value of a convex function are obtained by means of an approximating generalized moment problem. Numerical implementation is discussed in the context of stochastic programming problems.

1. Introduction: The generalized moment problem. The (generalized) moment problem: find $P : \mathscr{A} \to [0,1]$ such that

$$\int v_i(\xi) P(d\xi) \leq \beta_i, \qquad i = 1, \dots, s,$$

$$\int v_i(\xi) P(d\xi) = \beta_i, \qquad i = s+1, \dots, m, \quad \text{and} \tag{1.1}$$

$$z = \int v_0(\xi) P(d\xi) \text{ is maximized}$$

with $\mathscr{A}$ the sigma-field of events defined on Ξ—here a subset of R^N—is of general interest in statistics, in stochastic optimization, etc. The underlying premise is that some information is available about certain moments, or generalized moments, of an unknown probability distribution. This determines a class $\mathscr{P}$, i.e., the probability measures that satisfy the constraints of (1.1). This limited information is to be used in order to obtain an upper (or/and lower) bound in some other moment of the distributions in this class. Problem (1.1) has been studied in detail in the classical framework of statistical theory, see for example [12] where the accent is placed on those problems of type (1.1) that can be solved analytically (in the framework provided by Chebyshev systems). Conditions for feasible solutions were given in Kemperman [11]. Kemperman [10] also clarified the connection between the generalized moment problem and optimization theory but so far the computational tools provided by linear or nonlinear programming have only been used sparingly in the development of general solution procedures for generalized moment problems. This paper is one contribution in that direction.

Problem (1.1) can be especially useful in a stochastic programming problem such as

$$\text{find } x \in D \subset R^n \quad \text{such that}$$

$$z: c(x) + \int Q(x, \xi) F(d\xi) \text{ is minimized.} \tag{1.2}$$

*Received June 3, 1985; revised March 31, 1986.
AMS 1980 subject classification. Primary: 90C15.
IAOR 1973 subject classification. Main: Programming; Probabilistic.
OR/MS Index 1978 subject classification. Primary: 663 Programming/Stochastic.
Key words. Generalized moment problems, stochastic programming, inequalities in probability.
†This report has been partially supported by the National Science Foundation.
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0364-765X/87/1104/0149$01.25

In (1.2), the *recourse function* $Q(x, \xi)$ involves another optimization problem that renders the integration extremely difficult. Problem (1.1) can be used, however, to develop an upper bounding approximation of (1.2). In this case, F is replaced by a solution P of (1.1) in which the constraints determine some of the properties satisfied by F. Integration with respect to P is generally much simpler (involving, for example, a finite sum) than integration with respect to F.

Dupačová [4, 5] was the first to rely on some (classical) results for the moment problem to obtain bounds for the optimal value of certain stochastic programming problems; the class $\mathscr{P}$ is usually determined by first—or possibly second order—moments and v_0 is a 1-dimensional convex or concave nondifferentiable function. In such cases it is possible to obtain an explicit characterization of the extremal measure that solves the corresponding moment problem. This is not the case in general, although it is known that if the support of the probability measures Ξ is convex compact, then the linear programming problem (1.1) admits as optimal solution an extremal measure whose support is concentrated on at most $m + 1$ points of Ξ (see [1, Theorem 6.9] for a constructive proof of this result). There is usually no closed form expression that allows us to easily identify this extremal measure. There are no conceptual difficulties in designing solution procedures for solving (1.1), see [1, § 6]; however some of the operations that must be carried out may be in practice extremely onerous. Much depends on the properties of the functions v_0 and v_i, $i = 1, \ldots, m$. In this paper we shall be mostly concerned with the case when v_0 is convex, and the functions v_i, $i = 1, \ldots, m$ are linear or piecewise linear. We extend and sharpen the results of Dupačová [3] who works with bounded polyhedral support for the random variable ξ, and of Gassman and Ziemba [6] who allow for unbounded polyhedral support but of a restricted type. They also develop an implementable procedure for calculating this bound in the special, but important, case when $v(0)$ is convex and there is only one constraint on the expectation of ξ (with respect to P). By considering piecewise linear constraints and piecewise linear approximations of nonlinear constraints, we can obtain sharper bound on problems (1.2). We still solve only linear optimization problems and, thereby, avoid the difficulties of the nonlinear approaches in [1] and [2].

2. Linear constraints. Most of the literature devoted to generalized moment problems works with the assumption that Ξ is compact [9], [1, § 6]. Here we drop this assumption. It is possible to do so by relying on an appropriate compactification of R^n, viz. by adding to R^n the space of directions of recession.

Let us suppose that $\Xi \subset R^N$ is a nonempty closed convex set. If not, we could always work with the closure of co Ξ, its convex hull. Let

rc Ξ := cone of the directions of recession of Ξ.

This corresponds to the largest (closed) convex cone such that

$$\xi + (\mathrm{rc}\,\Xi) \subset \Xi \quad \text{for all } \xi \text{ in } \Xi;$$

it is called the *recession cone* of Ξ. Since $0 \in \mathrm{rc}\,\Xi$, it is never empty. For more about recession cones, consult [10, § 8].

Similarly, we allow ext Ξ to be any collection of points that contains the extreme points of Ξ. If Ξ has no extreme points, which means that the lineality space L of Ξ (as well as Ξ itself) contains lines, then ext Ξ must contain the extreme points of a polyhedral set (whose set of extreme points is never empty) obtained by intersecting Ξ with a linear space complementary to the lineality space of Ξ. As in the example that follows we shall only use the minimal number of points required for generating Ξ, but

that does not need to be the case. Consider, for example, $\Xi = \{(\xi_1, \xi_2) : \xi_1 = 1, \xi_2 \in R\}$. Here, we can let ext rc $\Xi = \{(0, 1), (0, -1)\}$ and ext $\Xi = \{(1, 1)\}$.

Let ext-rc Ξ be a collection of points such that rc Ξ = pos(ext-rc Ξ), i.e., the points positively span the (convex) recession cone. Note that if the lineality space of Ξ is nontrivial (i.e., not reduced to $\{0\}$), then ext-rc = Ξ will contain a positive basis for this lineality space, see [13] for example. It is usually possible to choose the elements of ext-rc Ξ so that they are positively linearly independent, in which case one can really think of the elements of ext-rc Ξ as extremal, but for our purposes this is not necessary.

Since Ξ is convex, we have

$$\Xi = \mathrm{co}(\mathrm{ext}\,\Xi) + \mathrm{pos}(\mathrm{ext} - \mathrm{rc}\,\Xi).$$

Every point ξ in Ξ has at least one representation of the form:

$$\xi = \int_{\mathrm{ext}\,\Xi} e\lambda(\xi, de) + \int_{\mathrm{ext}-\mathrm{rc}\,\Xi} r\mu(\xi, dr) \tag{2.1}$$

with $\lambda(\xi, \cdot)$ a probability measure on (ext Ξ, $\mathscr{E}$) and $\mu(\xi, \cdot)$ a nonnegative measure on (ext-rc Ξ, $\mathscr{R}$); $\mathscr{E}$ and $\mathscr{R}$ are the Borel fields. In fact, the theorems of Carathéodory and Steinitz guarantee that for each ξ there exists one representation involving no more than $N + 1$ points of ext Ξ and $2N$ points of ext-rc Ξ, i.e. the measures $\lambda(\xi, \cdot)$ and $\mu(\xi, \cdot)$ have then finite support.

Now, suppose v_0: $\Xi \to \bar{R}$ is a *lower semi-continuous* (has a closed epigraph) and *proper* (nowhere $-\infty$, and finite-valued somewhere) convex function. Consider ξ as given by (2.1) with

$$\hat{\xi} := \int_{\mathrm{ext}-\mathrm{rc}\,\Xi} r\mu(\xi, dr),$$

then

$$\xi = \int_{\mathrm{ext}\,\Xi} (e + \hat{\xi})\lambda(\xi, de), \quad \text{and} \tag{2.2}$$

$$v_0(\xi) \leq \int_{\mathrm{ext}\,\Xi} v_0(e + \hat{\xi})\lambda(\xi, de). \tag{2.3}$$

By

$$\mathrm{rc}\, v_0(\zeta) := \lim_{t \to \infty} \frac{v_0(\xi + t\zeta) - v_0(\xi)}{t} = \sup_{t > 0} \frac{v_0(\xi + t\zeta) - v_0(\xi)}{t} \tag{2.4}$$

we denote the *recession function* of v_0 in the direction ζ [14, § 8]. This is a sublinear function (positively homogeneous and convex). Since v_0 is convex, the ratio $t^{-1}(v_0(\xi + t\zeta) - v_0(\xi))$ is a monotone nondecreasing function of t (when $t > 0$) and thus the supremum is "attained" at $t = \infty$. This justifies the equivalence of the two formulas in (2.4). It also means that for all $e \in \mathrm{ext}\,\Xi$,

$$v_0(e + \hat{\xi}) \leq v_0(e) + \mathrm{rc}\, v_0(\hat{\xi})$$

and hence (2.3) yields

$$v_0(\xi) \leq \int_{\mathrm{ext}\,\Xi} v_0(e)\lambda(\xi, de) + \mathrm{rc}\, v_0(\hat{\xi}). \tag{2.5}$$

Since rc v_0 is a sublinear function, this implies that

$$v_0(\xi) \leq \int_{\mathrm{ext}\,\Xi} v_0(e)\lambda(\xi, de) + \int_{\mathrm{rc-ext}\,\Xi} \mathrm{rc}\, v_0(r)\mu(\xi, dr). \tag{2.6}$$

Integrating on both sides with respect to P, we obtain

$$\int v_0(\xi)P(d\xi) \leq \int_{\mathrm{ext}\,\Xi} v_0(e)\lambda(de) + \int_{\mathrm{rc-ext}\,\Xi} \mathrm{rc}\, v_0(r)\mu(dr) \quad \text{where} \tag{2.7}$$

$$\lambda(\cdot) = \int_\Xi \lambda(\xi, \cdot)P(d\xi) \quad \text{and} \tag{2.8}$$

$$\mu(\cdot) = \int_\Xi \mu(\xi, \cdot)P(d\xi), \tag{2.9}$$

(assuming that λ and μ have been chosen measurable with respect to ξ). Moreover

$$\bar{\xi} := \int \xi P(d\xi) = \int_{\mathrm{ext}\,\Xi} e\lambda(de) + \int_{\mathrm{ext-rc}\,\Xi} r\mu(dr), \tag{2.10}$$

$$1 = \int \lambda(de) = \int_\Xi P(d\xi)\int_{\mathrm{ext}\,\Xi} \lambda(\xi, de) = \int 1P(d\xi) \tag{2.11}$$

and both λ (defined on $\mathscr{E}$) and μ (defined on $\mathscr{R}$) are nonnegative.

The inequality (2.7) holds for any nonnegative measures λ and μ that satisfy (2.8)-(2.11) and (2.1). Note that (2.7) is nontrivial only if $\mathrm{rc}\, v_0(r) < +\infty$ for all $r \in$ rc-ext Ξ (as in the linear growth restriction in [6]). Given the bound provided by (2.7) one may be tempted to ignore (2.8) and (2.9)—i.e., that it must be possible to "disintegrate" λ and μ in measures $\lambda(\xi, \cdot)$ and $\mu(\xi, \cdot)$ that satisfy (2.1)—but this would render (2.7) invalid as an easy example will show readily. However, there always exists one pair (λ, μ) for which it is not necessary to verify if λ and μ can be "disintegrated" so as to satisfy (2.1): namely, if (λ, μ) maximizes the right-hand side of (2.7)! We have thus shown

THEOREM 2.1. *Suppose $\Xi \subset R^N$ is a nonempty closed convex set, P is a probability measure on $(\Xi, \mathscr{A})$, and v_0 is a lower semi-continuous, proper extended-real valued convex function defined on R^N. Then*

$$\int v_0(\xi)P(d\xi) \leq \sup_{(\lambda,\mu)} \int_{\mathrm{ext}\,\Xi} v_0(e)\lambda(de) + \int_{\mathrm{ext-rc}\,\Xi} \mathrm{rc}\, v_0(r)\mu(dr) \tag{2.12}$$

where λ is a probability measure on (ext Ξ, $\mathscr{E}$), and μ is a nonnegative measure on (ext-rc Ξ, $\mathscr{R}$) such that

$$\int_{\mathrm{ext}\,\Xi} e\lambda(de) + \int_{\mathrm{ext-rc}\,\Xi} r\mu(dr) = \int \xi P(d\xi) = \bar{\xi}. \tag{2.13}$$

This yields an upper bound on the optimal value of (1.1) when the constraints are determined by the linear system $\int \xi P(d\xi) = \bar{\xi}$. It should be emphasized that (2.12) yields in some sense the worst possible bound for $\int v_0(\xi)P(d\xi)$ among all those generated by (2.7)–(2.11), but it may be the only one that is sufficiently easy to compute and to integrate into an approximation scheme for solving stochastic optimization problems. The next result goes even further in that direction. It sharpens and extends [3, Theorem 2], [4, Theorem 4] or [1, § 5(v)] for the case of bounded convex polyhedral support.

COROLLARY 2.2. *Suppose $\Xi \subset C = \mathrm{co}(e^1, \ldots, e^p) + \mathrm{pos}(r^1, \ldots, r^q)$ and h: $R^N \to \bar{R}$ is a lower semicontinuous, proper convex function such that $h \geq v_0$ on Ξ. Then*

$$\int v_0(\xi)P(d\xi) \leq \sup_{(\lambda,\mu)}\left[\sum_{j=1}^p \lambda_j h(e^j) + \sum_{i=1}^q \mu_i(\mathrm{rc}\, h)(r^i) \,\middle|\, \lambda_j \geq 0, \mu_i \geq 0, \sum_{j=1}^p \lambda_j e^j + \sum_{i=1}^q \mu_i r^i = \bar{\xi}, \sum_{j=1}^p \lambda_j = 1\right]. \tag{2.14}$$

PROOF. Note that

$$\int v_0(\xi)P(d\xi) \leq \int_\Xi h(\xi)P(d\xi) = \int_C h(\xi)P(d\xi)$$

where P has been (trivially) extended to C. It now suffices to apply Theorem 2.1. ∎

All the results and remarks of this section apply equally well to the case when P is simply a bounded measure, in particular to the case when P is the restriction of some probability to a subset of the space of events, making of course the obvious adjustments. This simple observation yields directly the versions of Theorem 2.1 and Corollary 2.2 with conditional expectations.

3. Piecewise linear constraints. The extension to the case when the constraints are piecewise linear, or more precisely piecewise affine, is straightforward. Details are worked out in this section. The importance of this case rests on the potential use of piecewise linear approximations for handling the (general) nonlinear case, see § 4 for an elementary example.

Suppose Ξ is partitioned in L subregions Ξ_l, $l = 1, \ldots, L$, and in (1.1) the functions v_i, which define the constraints, are piecewise linear: for $i = 1, \ldots, m$,

$$v_i(\xi) = a_{il} \cdot \xi - \alpha_{il} \quad \text{when } \xi \in \Xi_l. \tag{3.1}$$

We then have

$$\int v_i(\xi)P(d\xi) = \sum_{l=1}^L \int_{\Xi_l} (a_{il}\xi - \alpha_{il})P(d\xi) \tag{3.2}$$

$$= \sum_{l=1}^L (a_{il}\bar{\xi}^l - \alpha_{il})p_l \quad \text{where} \tag{3.3}$$

$$\bar{\xi}^l = E\{\xi | \xi \in \Xi_l\}, \quad p_l = P(\Xi_l).$$

Thus, the (generalized) moment problem becomes:

$$\text{find } Q\colon \mathscr{A} \to [0,1], \text{ a probability measure, such that} \tag{3.4}$$

$$\bar{\xi}^l = E_Q\{\xi \mid \xi \in \Xi_l\}, \quad p_l = Q(\Xi_l),$$

$$\sum_{l=1}^{L} (p_Y a_{il})\bar{\xi}^l - \sum_{l=1}^{L} p_l \alpha_{il} \leqslant \beta_i, \qquad i = 1, \ldots, s,$$

$$\sum_{l=1}^{L} (p_l a_{il})\bar{\xi}^l - \sum_{l=1}^{L} p_l \alpha_{il} = \beta_i, \qquad i = s+1, \ldots, m,$$

$$\text{and } x = \int v_0(\xi) Q(d\xi) \text{ is maximized.}$$

Our objective is to obtain an upper bound on the optimal value of this problem. Let P be any probability measure that satisfies the constraints of (3.4); P could be the measure that we are trying to approximate.

Let C_l, $l = 1, \ldots, L$, be a collection of (nonempty) convex sets such that for all l, $\Xi_l \subset C_l$. One possibility is to choose for all $l = 1, \ldots, L$, $C_l = \mathrm{cl}(\mathrm{co}\,\Xi) :=$ the closure of the convex hull of Ξ. Let $\lambda'_l(\xi, \cdot)$ be probability measures defined on $(\mathrm{ext}\, C_l, \mathscr{E}_l)$ and $\mu'_l(\xi, \cdot)$ be nonnegative measures on $(\text{ext-rc}\, C_l, \mathscr{R}_l)$ such that for all $\xi \in \Xi_l$

$$\xi = \int_{\mathrm{ext}\, C_l} e \lambda'_l(\xi, de) + \int_{\text{ext-rc}\, C_l} r \mu'_l(\xi, dr). \tag{3.5}$$

Here $\mathscr{E}_l$ ($\mathscr{R}_l$ resp.) is the Borel field on $\mathrm{ext}\, C_l$ ($\text{ext-rc}\, C_l$ resp.) and we assume that λ'_l and μ'_l are $\mathscr{A}$-measurable with respect to ξ. For any $A \in \mathscr{E}_l$, set

$$\lambda_l(A) := \int_{\Xi_l} \lambda'_l(\xi, A) P(d\xi) \tag{3.6}$$

and for any $B \in \mathscr{R}_l$, set

$$\mu_l(B) := \int_{\Xi_l} \mu'_l(\xi, B) P(d\xi). \tag{3.7}$$

We have from (3.6):

$$\int_{\mathrm{ext}\, C_l} \lambda_l(de) = \lambda_l(\mathrm{ext}\, C_l) = P(\Xi_l) = p_l. \tag{3.8}$$

From (3.2), after replacing ξ by its representation (3.5), and interchanging the summands using the definitions of λ_l and μ_l, we obtain:

$$\int v_i(\xi) P(d\xi) = \left(\sum_{l=1}^{L} \int_{\mathrm{ext}\, C_l} a_{il} e\, \lambda_l(de) + \int_{\text{ext-rc}\, C_l} a_{il} r\, \mu_l(dr) - \alpha_{il} p_l \right). \tag{3.9}$$

From the above, by the same arguments as in § 2, in particular, by the convexity of v_0 on C_l, we prove the following generalization of Theorem 2.1.

THEOREM 3.1. *Suppose $\Xi \subset R^N$ is nonempty, $\{\Xi_l, l = 1, \ldots, L\}$ a partition of Ξ, $\{C_l, l = 1, \ldots, L\}$ nonempty, closed, convex sets such that $\Xi_l \subset C_l$ for all l, P a probability measure, v_0 an extended real-valued lower semicontinuous, proper convex function on R^N, and for $i = 1, \ldots, m$, v_i is a piecewise affine function on R^N with*

$$v_i(\xi) = a_{il}\xi - \alpha_{il} \quad \text{when } \xi \in \Xi_l.$$

Then

$$\int v_0(\xi) P(d\xi) \leqslant \sup_{(\lambda_l, \mu_l)} \sum_{l=1}^{L} \left[\int_{\mathrm{ext}\, C_l} v_0(e) \lambda_l(de) + \int_{\text{ext-rc}\, C_l} (\mathrm{rc}\, v_0)(r) \mu_l(dr) \right] \tag{3.10}$$

where for $l = 1, \ldots, L$, λ_l and μ_l are nonnegative measures on $(\mathrm{ext}\, C_l, \mathscr{E}_l)$ *and* $(\text{ext-rc}\, C_l, \mathscr{R}_l)$ *respectively, such that*

$$\lambda_l(\mathrm{ext}\, C_l) = P(\Xi_l) := p_l \tag{3.11}$$

and for $i = 1, \ldots, m$,

$$\sum_{l=1}^{L} \left(\int_{\mathrm{ext}\, C_l} a_{il} e \lambda_l(de) + \int_{\text{ext-rc}\, C_l} a_{il} r \mu_l(dr) \right) = \sum_{l=1}^{L} (p_l a_{il}) \bar{\xi}_l. \tag{3.12}$$

Note that moment conditions in (3.4) are introduced via the restrictions (3.11) and (3.12); P is only determined up to knowing the p_l and $\bar{\xi}_l$ associated with the region Ξ_l.

Observe also that no convexity conditions are necessary on the functions v_i, $i = 1, \ldots, m$. Suppose, for example with $N = 1$, that $a_{ik} = 1$, $a_{il} = 0$ if $l \neq k$, the ith condition of (3.12) would require that the measures be chosen so that

$$\int_{\mathrm{ext}\, C_k} e \lambda_k(de) + \int_{\text{ext-rc}\, C_k} r \mu_k(dr) = p_k \bar{\xi}_k.$$

This means that the measures on the extremal structure of C_k must satisfy these conditional expectation conditions. Approximation schemes can be built by requiring that the chosen measures satisfy conditional expectation conditions that involve finer and finer partitions of Ξ. Kall and Stoyan [8] and Huang, Vertinsky, and Ziemba [7] have studied constraints of that type. The constraints (3.7), however, are much more general, in that they allow for tighter restrictions so that sharper bounds may be obtained. The approximations in [7], [8] rely on the extreme points of the Ξ_l for all $l = 1, \ldots, L$, and therefore require the evaluation of the v_i for each of these points. Here only the extreme points of some convex set containing Ξ_l are needed. It is possible, for example, to choose $\Xi_l = \mathrm{co}\,\Xi$ for all $l = 1, \ldots, L$. Further restriction of C_l to subsets of Ξ, however, is advisable since this restricts the set of feasible probability measures, hence, improving the bounds.

There is also in the piecewise affine case a generalization of Corollary 2.2. We do not need v_0 convex, only that it be dominated by a convex function.

COROLLARY 3.2. *Suppose $\Xi_l \subset C_l := \mathrm{co}(e_l^1, \ldots, e_l^{r_l}) + \mathrm{pos}(r_l^1, \ldots, r_l^{q_l})$, $\{\Xi_l, l = 1, \ldots, L\}$ is a partition of Ξ, and $h\colon R^N \to \bar{R}$ is a lower semicontinuous, proper convex function such that $h \geqslant v_0$ on Ξ. Then for any probability measure $P\colon \mathscr{A} \to [0,1]$, we have*

$$\int v_0(\xi) P(d\xi) \leqslant \sup_{(\lambda, \mu)} \left[\sum_{l=1}^{L} \left(\sum_{k=1}^{r_l} h(e_l^k) \lambda_{lk} + \sum_{k=1}^{q_l} (\mathrm{rc}\, h)(r_l^k) \mu_{lk} \right) \right] \tag{3.13}$$

such that

$$\sum_{l=1}^{L}\left(\sum_{k=1}^{r_l} a_{il}e_l^k\lambda_{lk} + \sum_{k=1}^{q_l} a_{il}r_l^k\mu_{lk}\right) = \sum_{l=1}^{L} p_l a_{il}\bar{\xi}^l \quad \textit{for } i = 1,\dots,m,$$

$$\sum_{k=1}^{r_l}\lambda_{lk} = p_l, \qquad l = 1,\dots,L,$$

$$\lambda_{lk} \geq 0, \quad \mu_{lk} \geq 0,$$

where for $l = 1,\dots,L$, $p_l := P(\Xi_l)$ *and* $\bar{\xi}^l$ *is the conditional expectation (with respect to* P*) given that* $\xi \in \Xi_l$.

Of course to obtain this upper bound on $\int v_0(\xi)P(d\xi)$ it is not necessary to know P. It suffices to know the values of p_l and $\bar{\xi}^l$. In fact we do not even need the individual values of the $\bar{\xi}^l$. It is only necessary to know for each $i = 1,\dots,m$, $\alpha_i := \sum_{l=1}^{L} p_i a_{il}\bar{\xi}^l$.

More generally, suppose it is known that

$$\alpha_i^- \leq \int v_i(\xi)P(d\xi) \leq \alpha_i^+$$

but the precise value of this integral (with respect to P) is not known. In this case the constraint (3.12) could be replaced by

$$\alpha_i^- \leq \sum_{l=1}^{L}\left(\int_{\text{ext } C_l} a_{il}e\lambda_l(de) + \int_{\text{ext}-\text{rc } C_l} a_{il}r\mu_l(dr)\right) \leq \alpha_i^+. \tag{3.14}$$

Relation (3.14) is used when the generalized moment problem involves inequalities and the v_i are piecewise linear. This yields immediately an extension of Theorem 3.1 (to the inequality case) that turns out to be quite useful when dealing with higher order moment approximations. If it is known for example, that

$$\int \xi^2 P(d\xi) \leq \alpha_i, \tag{3.15}$$

then by defining a piecewise affine (lower) approximation v_i such that $v_i(\xi) \leq \xi^2$, the optimization problem (3.10) with the constraint (3.14)—defining $\alpha_i^- := 0$ and $\alpha_i^+ = \alpha_i$ —yields an upper bound on all probability measures satisfying the second moment condition (3.15). The bound can be improved by refining the approximation v_i.

For other than piecewise linear functions, limited results are still available. Suppose for example v_i is concave and

$$\int v_i(\xi)P(d\xi) \leq \alpha_i. \tag{3.16}$$

We can substitute for ξ its representation (3.5) to obtain

$$\sum_{l=1}^{L}\left(\int_{\Xi_l} v_i\left(\int_{\text{ext } C_l} e\lambda'(\xi, de) + \int_{\text{ext-rc } C_l} r\mu'(\xi, dr)\right)\right)P(d\xi) \leq \alpha_i.$$

Now if we use the concavity of v_i and the definitions (3.6) of λ_l and (3.7) of μ_l, we have

$$\sum_{l=1}^{L}\left(\int_{\text{ext } C_l} v_i(e)\lambda_l(de) + \int_{\text{ext-rc } C_l} \text{rc } v_i(r)\mu_l(dr)\right) \leq \alpha_i. \tag{3.17}$$

Therefore we can replace the constraint (3.16) by (3.17) and again obtain an upper bound on the expectation of v_0 that satisfies (3.16). Obviously the same technique can also be used if v_i is convex and we know a lower bound on the expectation of v_i.

4. Examples. The results of § 3 can be used to obtain bounds for a variety of lower dimensional cases. In this section we consider two piecewise affine approximations of the second moment function, $v_i(\xi) = \xi^2$. We take ξ to be a 1-dimensional random variable distributed on the interval $[\beta_1, \beta_2]$ with $\beta_1 < 0 < \beta_2$. Let $P([\beta_1, 0])$ $= p_1$, $P((0, \beta_2]) = p_2$ and suppose $P\{0\}) = 0$.

As upper approximate we define

$$v_i(\xi) = \begin{bmatrix} -\gamma_1\xi, & \xi \leq 0, \\ \gamma_2\xi, & \xi \geq 0, \end{bmatrix} \tag{4.1}$$

with $\gamma_1, \gamma_2 \geq 0$. If $\gamma_1 = \gamma_2$, then $v_i(\xi) = |\xi|$. In general we define γ_1 and γ_2 so as to best fit the case at hand, see Figure 1. With $C = \text{co}(\beta_1 = e^1, \beta_2 = e^2)$, the linear program (3.13) becomes:

find $\lambda_{11} \geq 0$, $\lambda_{12} \geq 0$, $\lambda_{21} \geq 0$, $\lambda_{22} \geq 0$ such that

$$\begin{aligned} -\gamma_1\beta_1\lambda_{11} - \gamma_1\beta_2\lambda_{12} + \gamma_2\beta_1\lambda_{21} + \gamma_2\beta_2\lambda_{22} &= \alpha, \\ \lambda_{11} + \lambda_{12} \qquad\qquad &= p_1, \\ \lambda_{21} + \lambda_{22} &= p_2, \end{aligned} \tag{4.2}$$

and $w = \sum_{l=1}^{2}\sum_{k=1}^{2} h(\beta_k)\lambda_{lk}$ is maximized. We set

$$\alpha := \int_{\beta_1}^{\beta_2} v_i(\xi)P(d\xi) = -\gamma_1\int_{\beta_1}^{0}\xi P(d\xi) + \gamma_2\int_{0}^{\beta_2}\xi P(d\xi) = -\gamma_1 p_1\bar{\xi}_1 + \gamma_2 p_2\bar{\xi}_2$$

where $\bar{\xi}_1$ and $\bar{\xi}_2$ denote conditional expectation with respect to $[\beta_1, 0]$ and $(0, \beta_2]$ respectively. Thus we can view (4.2) as finding a probability that assigns weights to the extreme points of $[\beta_1, \beta_2]$, viz. $(\lambda_{11} + \lambda_{21})$ to β_1 and $(\lambda_{12} + \lambda_{22})$ to β_2 in such a way that a certain generalized conditional expectation condition is satisfied.

There are four possible bases—each one corresponding to having 1 variable λ_{lk} nonbasic—depending on the values of the coefficients. The following table gives the solution values and optimality conditions when $\gamma_1 = \gamma_2 > 0$, and for 2 of the 4 possible

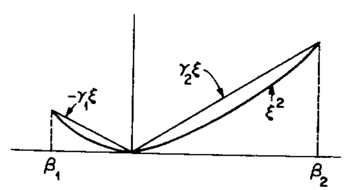

FIGURE 1. Upper "Absolute Value" Approximate.

bases (the other two have λ_{21} and λ_{22} nonbasic respectively):

TABLE 1

Optimality Conditions: $\gamma_1 = \gamma_2 > 0$

Nonbasic Variable	λ_{11}	λ_{12}
Dual feasibility	$h(\beta_2) \geqslant h(\beta_1)$	$h(\beta_2) \leqslant h(\beta_1)$
Primal feasibility	$p_1 \leqslant p_2(\frac{\beta_2 - \bar{\xi}_2}{\beta_2 - \bar{\xi}_1})$	$p_1 \leqslant p_2(\frac{\beta_1 - \bar{\xi}_2}{\beta_1 - \bar{\xi}_1})$
Solution:	$\lambda_{12} = p_1$ $\lambda_{21} = p_2 - \lambda_{22}$ $\lambda_{22} = \frac{1}{\beta_2 - \beta_1}[[p_1(\bar{\xi}_1 - \beta_2) + p_2(\beta_1 - \bar{\xi}_2)]]$	$\lambda_{22} = p_1$ $\lambda_{12} = p_1 - \lambda_{11}$ $\lambda_{11} = \cdots$

The dual feasibility conditions are the same for all γ_1 and γ_2 positive. The primal feasibility conditions depend on the relative sizes of the slopes.

We can also work with a lower approximate for $\xi \mapsto \xi^2$, for example with the cup-shaped function:

$$v_i(\xi) = \begin{cases} \gamma_1(\beta_{11} - \xi) & \text{if } \beta_1 \leqslant \xi \leqslant \beta_{11}, \\ 0 & \text{if } \beta_{11} \leqslant \xi \leqslant \beta_{21}, \\ \gamma_3(\xi - \beta_{21}) & \text{if } \beta_{21} \leqslant \xi \leqslant \beta_2, \end{cases}$$

where $\beta_1 < \beta_{11} < 0 < \beta_{21} < \beta_2$. See Figure 2. If v_i is this cup-shaped function, the associated linear program (that yields an upper bound) reads:

find $\lambda_{11} \geqslant 0, \lambda_{12} \geqslant 0, \ldots, \lambda_{32} \geqslant 0$ such that

$$-\gamma_1\beta_1\lambda_{11} - \gamma_1\beta_2\lambda_{12} + \gamma_3\beta_1\lambda_{31} + \gamma_3\beta_2\lambda_{32} = \alpha.$$

$$\lambda_{l1} + \lambda_{l2} = p_l, \quad l = 1, 2, 3, \quad \text{and}$$

$$w = \Sigma_{l=1}^{3}\Sigma_{k=1}^{2} h(\beta_k)\lambda_{lk} \textit{ is maximized}, \quad \textit{where}$$

$$\alpha := -p_1\gamma_1\bar{\xi}_1 + p_3\gamma_3\bar{\xi}_3.$$

There are 8 feasible bases. The solutions are of the same type as in Table 1.

The solution of (4.3) is useful in conjunction with the solution of (4.2). As we noted earlier, if $v_i(\xi) \leqslant \xi^2$ then the solution of (3.13) yields an upper bound on $\int v_0(\xi)P(d\xi)$. If, however, $v_i(\xi) \geqslant \xi^2$, then the solution of (3.13) yields a lower bound on the supremum over all distributions satisfying the second moment condition. The two

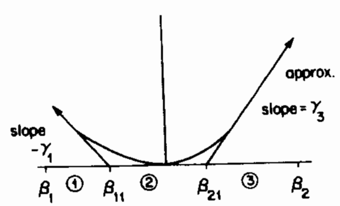

FIGURE 2. Cup-Shaped Approximation for ξ^2.

bounds can be used to determine how well the second moment condition has been approximated.

5. Applications in stochastic programming. Bounds on the expectation of a convex function can be particularly useful in the solution of stochastic optimization problems. In these problems, it is often necessary to perform an optimization to evaluate a function at a single point. Taking the expectation of such functions presents a formidable task. By limiting the support of the distribution of a finite number of points, as in the development above, the set of problems to optimize is limited, and one can efficiently obtain bounds on the expectation.

A typical stochastic optimization problem has the following form:

$$\inf_{x \in K} c(x) + \int_{\Xi} Q(x, \xi)P(d\xi), \tag{5.1}$$

where $Q(x, \xi)$ is itself the optimal value of a mathematical program whose coefficients depend on x and ξ. To solve (5.1), we need to evaluate $\int Q(x, \xi)P(d\xi)$ for many values of x chosen in an optimization procedure. An upper bound on the integral that is easily calculable may lead to useful bounds on the optimal value of (5.1).

As an example, consider

$$\begin{aligned} Q(x, \xi) = \min \quad & 5y_1 + 10y_2 + 10y_3 \\ \text{s.t.} \quad & y_1 + y_2 = \xi_1 - x_1, \\ & y_1 + y_3 = \xi_2 - x_2, \\ & y_1, y_2, y_3 \geqq 0. \end{aligned} \tag{5.2}$$

Note that this problem has the values

$$Q(x, \xi) = \begin{cases} 10(\xi_1 - x_1) - 5(\xi_2 - x_2) & \text{if } \xi_1 - x_1 \geqslant \xi_2 - x_2 \geqslant 0, \\ 10(\xi_2 - x_2) - 5(\xi_1 - x_1) & \text{if } \xi_2 - x_2 \geqslant \xi_1 - x_1 \geqslant 0, \\ +\infty & \text{otherwise.} \end{cases}$$

We wish to consider the case where $x = (0, 0)$ and $P[\xi_1 \leqslant \bar{\xi}_1, \xi_2 \leqslant \bar{\xi}_2] = \int_0^{\bar{\xi}_1}\int_0^{\bar{\xi}_2} 4e^{-2\xi_1}e^{-2\xi_2}d\xi_1 d\xi_2$, where $\bar{\xi}_1$ and $\bar{\xi}_2$ are any given nonnegative values of the random variables ξ_1 and ξ_2. In this way, ξ_1 and ξ_2 are independent exponential random variables with means $1/2$.

We wish to investigate how to find an upper bound on

$$\int Q((0, 0), \xi)P(d\xi) \tag{5.3}$$

without computing the integral exactly. We first note that the exact value is 6.25. The other methods we will try will be Gassman and Ziemba's approach, a solution using Corollary 3.2 and not exploiting the independent of the random variables, and a solution using Corollary 3.2 and the independence of the random variables.

A lower bound on (5.3) is found using Jensen's Inequality as $Q((0, 0), \bar{\xi}) = 2.5$. An upper bound is available using the Gassman-Ziemba approach which reduces to (3.13) with $a_{il} = e_l$ and $L = n$. Using the extreme directions of $\Xi = [0, \infty) \times [0, \infty)$ as $(1, 0)$ and $(0, 1)$, an extreme point of $(0, 0)$, and noting that $\text{rc}\, Q((0, 0), (1, 0)) = \text{rc}\, Q((0, 0),$

$(0,1)) = 10$, the following version of (3.13) is obtained,

$$\int Q((0,0),\xi)P(d\xi) \leqslant \sup_{(\lambda,\mu)} \quad 10\mu_1 + 10\mu_2$$
$$\text{s.t.} \quad \begin{aligned} 0\lambda_1 + \mu_1 \qquad\quad &= 1/2 \\ 0\lambda_1 \qquad\quad + \mu_2 &= 1/2 \\ \lambda_1 \qquad\qquad\quad &= 1 \\ \lambda_1, \quad \mu_1, \quad \mu_2 &\geqq 0. \end{aligned} \tag{5.4}$$

The only feasible solution to (5.4) is $\mu_1 = \mu_2 = 1/2$, $\lambda_1 = 1$, and the upper bound so obtained is 10. By using the independence of the random variables, we can see that, for any value $\bar{\xi}_2$ of ξ_2, $Q((0,0),(\xi_1,\bar{\xi}_2))$ defined on $[0,+\infty)$ has a value of $10\bar{\xi}_2$ at $(0,\bar{\xi}_2)$ and a recession function value of 10 in the direction of increasing ξ_1. A Gassman-Ziemba bound on the integral over ξ_1 is therefore, $10\bar{\xi}_2 + 5$ for any $\bar{\xi}_2$. Using this value, we can then solve the problem in ξ_2 and again obtain a value of 5 at $(0,0)$ and a recession function of 10. The resulting upper bound is again 10, so the use of independence does not improve the bound.

To provide a more precise bound on (5.3), we include more constraints in (1.1). Note that $E(\xi_1^2) = E(\xi_2^2) = 1/2$. A function $v_1(\xi_1,\xi_2)$ such that $v_i \leqslant \xi_i^2$ is

$$v_i(\xi) = \begin{cases} 0 & \text{if } \xi_i \leqslant 1/2, \\ 2\xi_i - 1 & \text{if } \xi_i \geqslant 1/2, \end{cases} \tag{5.5}$$

a cup-shaped function as in Figure 2. This can be used with bounds of 1/2 on $E(v_i(\xi))$ for $i = 1,2$ to obtain additional constraints to mean value constraints. The region of Ξ must then be partitioned into Ξ^1, Ξ^2, Ξ^3, and Ξ^4 as in Figure 3.

We use $h(\xi) = Q((0,0),\xi)$ and $C^l = \Xi^l$ with constraints on $v_l = e_{l-2}$, $l = 3$, 4, and v_i as in (5.5). The resulting linear program is:

$$\begin{aligned} \max \quad & 5\lambda_{12} + 5\lambda_{13} + 2.5\lambda_{14} + 5\lambda_{21} + 2.5\lambda_{22} \\ & + 10\mu_{21} + 5\lambda_{31} + 2.5\lambda_{32} + 10\mu_{31} + 2.5\lambda_{41} \\ & + 10\mu_{41} + 10\mu_{42} \end{aligned} \tag{5.6.0}$$

$$\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} = 0.3996, \tag{5.6.1}$$

$$\lambda_{21} + \lambda_{22} = 0.2325, \tag{5.6.2}$$

$$\lambda_{31} + \lambda_{32} = 0.2325, \tag{5.6.3}$$

$$\lambda_{41} = 0.1353, \tag{5.6.4}$$

$$\lambda_{13} + \lambda_{14} + \lambda_{22} + \lambda_{31} + \lambda_{32} + \lambda_{41} + 2\mu_{31} + 2\mu_{42} = 1, \tag{5.6.5}$$

$$\lambda_{12} + \lambda_{14} + \lambda_{21} + \lambda_{22} + \lambda_{32} + \lambda_{41} + 2\mu_{21} + 2\mu_{41} = 1, \tag{5.6.6}$$

$$2\mu_{31} + 2\mu_{42} \leqslant 0.5, \tag{5.6.7}$$

$$2\mu_{21} + 2\mu_{41} \leqslant 0.5, \tag{5.6.8}$$

$$\lambda_{ij}, \mu_{ij} \geqq 0 \quad \text{for all } i, j, \tag{5.6.9}$$

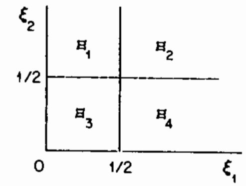

FIGURE 3. Partitions of Ξ.

where (5.6.1-4) corresponds to the constraint on $P\{\Xi_l\}$, (5.6.5-6) constrains $E(\xi)$ and (5.6.7-8) is an outer approximation to the second moment constraint (3.15). The linear program (5.6) has an optimal value of 8.98.

We can improve on this solution value by using the independence of ξ_1 and ξ_2. We consider first $\int Q((0,0),(\xi_1,0))P^1(d\xi_1)$ where P^1 is the marginal probability distribution of ξ_1. Clearly, this has a value of $10\bar{\xi}_1 = 5$ since $Q((0,0),(\xi_1,0))$ is linear. Next, consider $\int Q((0,0),(\xi_1,1/2))P^1(d\xi_1)$. For this problem, we use $v_1(\xi)$ as in (5.5) and an upper bound of 1/2 and solve the resulting program (3.13) in ξ_1 only. The result is an upper bound of 6.25. We note here that using the partition $\Xi_1^1 = [0,1/2]$, $\Xi_1^2 = (1/2,+\infty)$, without the constraint on $v_1(\xi)$ leads to an upper bound of 7.24. The restriction on the second moment therefore allows for a 14% improvement in the bound.

The bound on $\int Q((0,0),(\xi_1,1/2))P^1(d\xi_1)$ can then be used to bound (5.3). We use that 5 is a bound at $\bar{\xi}_2 = 0$, that 6.25 is a bound at $\bar{\xi}_2 = 1/2$ for $\int Q(0,0)$, $(\xi_1,\bar{\xi}_2)P^1(d\xi_1)$ and that the recession function value of $\int Q(0,0),(\xi_1,\bar{\xi}_2))P^1(d\xi_1)$ is 10. This yields another problem similar to (5.6) that can be solved with a constraint on $v_2(\xi)$ to yield an overall upper bound of 8.12. This represents a 10% improvement over the bound without using independence, and a 19% improvement over the bound which only considers mean values. The results are summarized in Table 2. We note that using mean values for each subset of the partition yields a bound of 7.53. This bound, however, requires the computation of conditional means on $\Xi_1,\ldots,\Xi_4$. The bound of 8.12 only requires knowledge of the overall first and second moments and $P\{\Xi_l\}$, $l = 1,\ldots,4$, that are more readily accessible.

This example provides one indication of the usefulness of including constraints on second moments in the solution of stochastic optimization problems. As a second example, we consider a linear support region and a nonlinear function v_0. Let

$$Q(x,\xi) = \begin{cases} (\xi - x)^2 & \text{if } |\xi - x| \leqslant 5, \\ 10(\xi - x) & \text{if } \xi - x \geqslant 5, \\ -10(\xi - x) & \text{if } \xi - x \leqslant 5, \end{cases} \tag{5.7}$$

where ξ is normally distributed. We wish to bound $\int Q(0,\xi)P(d\xi)$ where $\bar{\xi} = 0$ and

TABLE 2

Bounds on $\int Q((0,0),\xi)P(d\xi)$.

Constraints	Independent Computation	Upper Bound
Means	No	10.00
Means	Yes	10.00
Means & Second Moments	No	8.98
Means & Second Moments	Yes	8.12

$E(\xi^2) = 1/4$. We use $v_1 = \xi$ for the mean value constraint and

$$v_2(\xi) = \begin{cases} 2\xi - 1 & \text{if } \xi \geq 1/2, \\ 0 & \text{if } -1/2 \leq \xi \leq 1/2, \\ -2\xi - 1 & \text{if } \xi \leq -1/2, \end{cases} \tag{5.8}$$

where $v_2(\xi) \leq \xi^2$. The resulting linear program, with $\Xi^1 = (-\infty, -1/2]$, $\Xi^2 = (-1/2, 1/2]$, $\Xi^3 = (1/2, +\infty)$, and a bound of 1/4 on $\int v_2(\xi) P(d\xi)$ has an optimal value of 1.5. Without the bound on $\int v_2(\xi) P(d\xi)$, however, the linear program is unbounded because of the distribution's symmetry. In this case, it is essential to include additional piecewise linear constraints to the mean value constraint.

Acknowledgement. We gratefully acknowledge the helpful comments of Jitka Dupačová, Gus Gassmann, and the referees.

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MATHEMATICS OF OPERATIONS RESEARCH
Vol. 12, No. 1, February 1987
Printed in U.S.A.

ON THE CONVERGENCE OF POLICY ITERATION IN FINITE STATE UNDISCOUNTED MARKOV DECISION PROCESSES: THE UNICHAIN CASE*†

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We study the convergence of policy iteration for the undiscounted, finite state, discrete time Markov decision problem with compact action space and unichain transition structure. Using a "Newton Method type" representation for policy iteration, we establish the existence of a solution to the optimality equation. We show that to find an average optimal policy, it is sufficient to solve the optimality equation on the recurrent set of the maximizing policy. Under the additional assumption of a unique maximizing policy at each stage of the policy iteration procedure, we show that the iterates are convergent and the resulting policy is Blackwell optimal.

1. Introduction. In this paper we study the policy iteration procedure (PIP) in finite state, average reward Markov decision processes with compact action spaces. Its purpose is to extend the usual finite state and action results, Howard (1960), to a situation where the PIP is not finitely convergent. The motivation is work of Puterman and Brumelle (1978), (1979) on the relationship of the PIP to Newton's Method in discounted MDP's. Here, policy iteration requires studying both the gain and bias, and we develop "Newton type" representatives for each. Proofs of convergence are based on a two-stage argument where first the optimality equation is solved on the recurrent set of an average optimal policy and then on the transient states.

We briefly review the relevant literature on policy iteration in undiscounted Markov decision processes. Policy iteration for finite state and action undiscounted MDP's is usually attributed to Howard (1960). He showed that the PIP is convergent to an average optimal policy when all states are recurrent under all policies. Blackwell (1962) provided a rigorous framework for this problem based on the fundamental matrix, established the existence of bias-optimal policies using policy iteration and proved the existence of what are now called Blackwell-optimal policies. The computation of bias-optimal policies was studied by Veinott (1966) and Denardo (1970), (1973). In Miller and Veinott (1969) and Veinott (1969), a policy iteration method for finding n-discount optimal policies and Blackwell-optimal policies was developed.

Implementation of the PIP in the finite state and action case with multichain structure presents extra difficulties because without careful selection of the relative values, the algorithm may cycle. Schweitzer and Federgruen (1978) provided a specification for the choice of additive constants to ensure convergence. Federgruen and

*Received October 14, 1983; Revised September 9, 1985.
AMS 1980 subject classification. Primary: 90C39.
IAOR 1973 subject classification. Main: Programming: dynamic.
OR/MS Index 1978 subject classification. Primary: 117 Dynamic Programming/Markov.
Key words. Markov Decision Processes, Average Reward, Policy Iteration.
†This research was partially supported by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.) and the Natural Science and Engineering Research Council of Canada Grants A-3609 and A-5527.
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0364-765X/87/1104/0163$01.25