THE VALUE OF INFORMATION AND STOCHASTIC PROGRAMMING

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The problem of planning under uncertainty has many aspects; in this paper we consider the aspect that has to do with evaluating the state of information. We address ourselves to the question of how much better (i.e., how much more profitable) we could expect our plans to be if somehow we could know at planning time what the outcomes of the uncertain events will turn out to be. This expected increase in profitability is the 'expected value of perfect information' and represents an upper bound to the amount of money that it would be worthwhile to spend in any survey or other investigation designed to provide that information beforehand. In many cases, the amount of calculation to compute an exact value is prohibitive. However, we derive bounds (estimates) for the value. Moreover, in the case of operations planning by linear or convex programming, we show how to evaluate these bounds as part of a post-optimal analysis.

WE CONSIDER a situation in which the profit (or other measure of desirability) depends upon the decision as to which particular action we take, and the outcome of some random event. An example is given by the situation in which we must manufacture to meet market demands. The decision is how much to manufacture, the random event is the market demand that will occur, and the profit depends on both of these variables. Quite generally, let x be the decision variable and let z be the random variable. We write $\varphi(x, z)$ for the profit. Note that x and z may each be vectors.

We are interested in problems where the decision variable x must be chosen *first*, then the random variable is observed, and finally there is recourse to some contingency plan to tidy up. If we wish to maximize *expected* profit (if not, we replace φ by some other utility function), we choose x so as to maximize $E\varphi(x, z)$. We call this the *recourse problem* (*RP*), using a terminology similar to the one introduced by WALKUP AND WETS.[6] We write

$$RP = \max_x E\varphi(x, z) \tag{1}$$

as the maximum expected profit in such a situation.

If, however, we were able to reverse the order of events, i.e., if we could wait and see what random event would occur *before* we had to make a decision, then we could choose x so as to maximize $\varphi(x, z)$ for the z that actually occurs. The expected profit in this case, which we call, according to terminology introduced by A. MADANSKY,[3] the *wait-and-see* (WS) case, is therefore

$$WS = E\max_x \varphi(x, z). \tag{2}$$

It should be intuitively clear that the expected profit in the wait-and-see case is always at least as great as that in the recourse-problem case. Formally, it is easy to show (Theorem 1) that $WS \geqq RP$, with the above definitions, for *any* real function $\varphi(x, z)$ and for any probability distribution function, subject only to the existence of the maxima and expected values involved.

Given a choice, therefore, with other considerations equal, the wait-and-see situation is always a better one than the recourse-problem situation; in fact, it would generally be worthwhile to pay an amount up to $WS - RP$ to move from the recourse problem situation to the wait-and-see situation.

Consider, then, a decision maker with a recourse problem situation. He may ask whether it is worthwhile to buy information about the random variable that may influence his decisions. In some cases, this might mean making a market survey; in other cases, it might mean inducing the customers to commit themselves in advance to what their demands will be. In any case, he is asking how much money it would be worthwhile to spend in order to know what the value of the random variable is, so as to know it before he has to make his decision x. In buying information, then, he is changing his situation from the recourse one to the wait-and-see; the maximum amount that he would pay therefore would be the difference $WS - RP$. This difference has been called the *expected value of perfect information*, $EVPI$.[4, 5] We write

$$EVPI = WS - RP. \tag{3}$$

BOUNDS FOR THE EXPECTED VALUE OF PERFECT INFORMATION

AGAIN, LET $\varphi(x, z)$ be the profit as a function of the decision x and the random variable z. $E\varphi(x, z)$ is the expected profit for decision x. The recourse problem is then the problem in which x must be chosen before knowing the value of z, and the expected value of perfect information is, by (1), (2), and (3),

$$EVPI = E\max_x \varphi(x, z) - \max_x E\varphi(x, z). \tag{4}$$

THEOREM 1. *Let the expected values and the indicated maxima exist. Then* $EVPI \geqq 0$.

Proof. For every $\hat{x}$, and z, we have $\max_x \varphi(x, z) \geqq \varphi(\hat{x}, z)$. If we now take expectations on both sides and choose $\hat{x}$ so as to maximize $E\varphi(x, z)$, we obtain $EVPI = E\max_x \varphi(x, z) - E\varphi(\hat{x}, z) \geqq 0$.

While the result is obvious, and well known,[4] it deserves mention in order to emphasize that no assumptions other than the existence of expected values and maxima are involved. In particular, it is not necessary to assume that $\varphi(x, z)$ is concave.

Formula (4), which defines the expected value of perfect information, is not generally useful for calculation, since the WS term, i.e., $E\max_x \varphi(x, z)$, generally involves a prohibitive amount of calculation, sometimes even to approximate it. It requires, in general, that, for each possible z, the optimization problem $\max_x \varphi(x, z)$ be solved. To obtain more information on the problem of solving a wait-and-see program, the reader is referred to BEREANU[2] and to the references cited there.

If we assume that $\varphi(x, z)$ is a *concave* function, it is possible to compute a *bound* for the expected value of perfect information.

Let $\psi(z)$ be a concave function of the random variable (generally a vector) z. For any point z^0 at which ψ is differentiable, we may write

$$\psi(z) \leqq \psi(z^0) + \nabla\psi(z^0)(z - z^0). \tag{5}$$

Actually, even if ψ is not differentiable at z, this formula is valid, provided we interpret $\nabla\psi(z^0)$ as a vector, each component of which is any value less than or equal to the left partial directional derivative and greater than or equal to the right partial. Assuming existence of first moments, therefore,

$$E\psi(z) \leqq \psi(z^0) + \nabla\psi(z^0)(Ez - z^0). \tag{6}$$

Now let $\varphi(x, z)$ be a function concave in z, such that $\psi(z) = \max_x \varphi(x, z)$ exists for all z. Then $\psi(z)$ is concave, so from (6) we have

$$E\max_x \varphi(x, z) \leqq \max_x \varphi(x, z^0) + \nabla\psi(z^0)(Ez - z^0). \tag{7}$$

From (4) and Theorem 1,

$$0 \leqq EVPI \leqq \max_x \varphi(x, z^0) - \max_x E\varphi(x, z) + \nabla\psi(z^0)(Ez - z^0) \tag{8}$$

for any z^0.

Since (8) holds for any z^0, we may ask which z^0 will give the best estimate, in the sense of giving a tightest bound.

THEOREM 2. *The best possible bound for an estimate of the type* (8) *is obtained by choosing* $z^0 = Ez$.

Proof. We have to show that the function $F(z^0)$ defined by

$$F(z^0) = \max_x \varphi(x, z^0) + \nabla\psi(z^0)(Ez\text{-}z^0) = \psi(z^0) + \nabla\psi(z^0)(Ez - z^0)$$

has a minimum at $z^0 = Ez$. Since $\psi(z)$ is concave and is assumed to exist for all z, $\psi(Ez) \leqq \psi(z^0) + \nabla\psi(z^0)(Ez - z^0)$, so that $F(Ez) = \psi(Ez) \leqq F(z^0)$ for all z^0.

When z^0 is set equal to Ez in (8), we obtain

$$0 \leqq EVPI \leqq \max_x \varphi(x, Ez) - \max_x E\varphi(x, z).$$

The first term on the right-hand side is the maximum profit obtainable in a deterministic problem (EV), formed by replacing the random variable z by its expected value Ez. Then we have $EV = \max_x \varphi(x, Ez)$. Inequality (8) may be rewritten as

$$0 \leqq EVPI \leqq EV - RP. \tag{9}$$

The evaluation of this 'best' estimate requires, however, the calculation of an optimal solution of a stochastic problem (RP) and the calculation of an optimal solution of a deterministic problem (EV). We now develop a bound for the EVPI that, while it is not generally as tight as (8) or (9), does not require so much calculation.

Let $x(z^0)$ maximize $\varphi(x, z^0)$, i.e., let $x(z^0)$ satisfy $\varphi[x(z^0), z^0] = \max_x \varphi(x, z^0)$. Note that, for any z^0, $\max_x E\varphi(x, z) \geqq E\varphi[x(z^0), z]$. Combining with (8), therefore, we have

$$0 \leqq EVPI \leqq \varphi[x(z^0), z^0] - E\varphi[x(z^0), z] + \nabla\psi(z^0)(Ez - z^0). \tag{10}$$

By means of this formula, we may evaluate a bound on the expected value of information, and the only optimization problem we have solved is a deterministic one, formed by replacing the random variable by any value z^0, subject only to the proviso that the optimization calculation also yields the 'sensitivities' required to evaluate $\nabla\psi(z^0)$. We shall expand on this point in the next section.

APPLICATION TO STOCHASTIC PROGRAMMING

Stochastic Linear Programming

We first consider the case in which the manufacturing process is described by a linear activity analysis model, i.e., by a linear program, and in which the right-hand side, representing the demands, is an m-dimensional random vector. We assume the manufacturer's problem can be described by a linear program of the form

$$\min cx \quad \text{subject to} \quad Ax = z,\ x \geqq 0. \tag{11}$$

Here, z is the supply/demand vector, and the components of $y = Ax$ are the amounts, as a function of the decision vector x, of the various commodities produced (net after disposal of excess, if necessary), or used up.

If the demand is a random variable whose value is revealed only after x (and therefore Ax) have been determined, we cannot require $Ax = z$; rather, we define a *recourse* function g, which specifies an extra cost (or revenue) as a function of the difference $z - Ax$ or $z - y$. The problem of choosing x

so as to minimize expected cost, then, is a special case of the recourse problem of the first two sections, where

$$\varphi(x, z) = -cx - g(z - Ax). \tag{12}$$

We shall assume that the recourse function g is convex, so that $\varphi(x, z)$ is concave. We also assume $g(0) = 0$.

The recourse function is supposed to represent the cost of 'emergency' supply in case too little of a commodity is produced, or the revenue from an 'emergency' disposal market if too much is produced; therefore, we would not expect that the activity whose cost is given by g would be used at all if the demand z were known beforehand, even if that activity were available. Alternatively, we could say that if that activity were available at the time of planning x, it should be included in the A matrix, making the original problem nonlinear, but still convex.

Either way, the condition that the emergency supply or disposal possibility would not be used at the time of planning x is that for a particular value of z the dual solution π^* for the linear program (11) satisfy

$$\nabla g_-(0) \leqq \pi^* \leqq \nabla g_+(0), \tag{13}$$

where $\nabla g_-(0)$ and $\nabla g_+(0)$ are, respectively, the vectors of left and right partial derivatives of g at zero level. In the case of 'linear' recourse,[8] for example, we assume g is given by

$$g(z-y) = g(z-Ax) = \sum_i \begin{cases} \gamma_i(z-Ax)_i, & \text{if} \quad (z-Ax)_i \geqq 0, \\ \delta_i(z-Ax)_i, & \text{if} \quad (z-Ax)_i \leqq 0, \end{cases} \tag{14}$$

and the above condition is then just $\delta \leqq \pi^* \leqq \gamma$.

Let us describe how the estimates of the expected value of information could be made in practice. Imagine that an operations planner sets up a linear program of the type (11), on the basis of some definite demand vector z^0. He then obtains an optimal primal solution x^*, and an optimal dual solution π^*. That is, he follows the present procedures for setting up and solving a linear program.

As part of his post-optimal study, he then asks what if some of the demands should really have been considered as random variables. He then proposes a reasonable probability distribution function for the demand vector, and in addition, proposes a recourse function g. Suppose for this g, (13) is satisfied. Now, he asks, what would be a bound on the value of demand information, i.e., what would be an upper limit to the amount it would be worth while spending to find out beforehand what the demands would be?

Relation (10) may be used. The term $\varphi[x(z^0), z^0]$ is just $\varphi(x^*, z^0)$, which, in turn, is just $-cx^*-g(z^0-Ax^*)=-cx^*$, since $g(0)=0$. The second term $E\varphi(x^*, z)$ reduces to $-cx^*-Eg(z-Ax^*)$. To evaluate the third term, we use the fact that if the dual optimal solution is unique, then $-\pi^*=\nabla\psi(z^0)$; and if not, every dual solution has the property that $-\pi_i^*$ is between the right and the left partial derivative of $\psi(z^0)$.[7] Thus, in any case, $-\pi^*$ may be substituted for $\nabla\psi(z^0)$, thus obtaining $\pi^*z^0-\pi^*Ez$. Relation (10) is just

$$0\leqq EVPI\leqq Eg(z-Ax^*)-\pi^*(Ez-z^0), \tag{15}$$

which could easily be evaluated by hand or at most by a desk calculator if g is a reasonably simple function to evaluate.

More General Stochastic Programs

It should be evident that there is no reason to restrict our attention to the linear case; in fact if (11) were a convex nonlinear program, the analysis would be exactly the same, and relation (13) is valid if the π^* are interpreted as the Lagrange multipliers.

Generalization in the direction of several time stages is also possible in principle. Suppose that having chosen x (and thereby y, from $y=Ax$) and having observed z, the difference $z-y$ is to be made up by solving a second (possibly stochastic) optimization problem. If this second problem is again convex, its maximum profit will be a convex function of the quantity $z-y$, thereby defining the convex recourse function g. Generally, however, this g would be given by the wait-and-see problem connected with the second stage, and so would not lead to a convenient calculation. Nevertheless, in some special cases the calculation might be made, and of course, the possibility of again approximating by $WS\geqq EV$, or by some other form, should be taken into account.

Stochastic Programming with Quadratic Recourse

In case the recourse function g is smooth at zero, and can be represented satisfactorily by a convex quadratic function,[10] a particularly elegant form of an estimate for the value of information can be derived directly from (9). We suppose that g is given by

$$g(z-y)=q(z-y)+(z-y)'Q(z-y), \tag{16}$$

where $y=Ax$; q is a constant vector, Q is a positive semidefinite matrix, and the prime means transpose. Thus we have the case

$$\varphi(x, z)=-cx-[q(z-y)+(z-y)'Q(z-y)],$$

which represents the 'stochastic linear program with quadratic recourse.' To evaluate (9), we have (with $y \equiv Ax$)

$$EV = \max_{x \geqq 0} \{-cx - [q(Ez-y) + (Ez-y)'Q(Ez-y)]\}$$
$$= \max_{x \geqq 0} \{-cx - q(Ez-y) + 2y'QEz - y'Qy - (Ez)'QEz\},$$

and also

$$RP = \max_{x \geqq 0} \{-cx - E[q(z-y) + (z-y)'Q(z-y)]\}$$
$$= \max_{x \geqq 0} \{-cx - q(Ez-y) + 2y'QEz - y'Qy' - E(z'Qz)\},$$

from which we obtain by subtraction $EV - RP = E[z'Qz - (Ez)'QEz]$.

Every symmetric positive semidefinite (psd) matrix has a 'square root,' i.e., if Q is psd, there exists a matrix T such that $T'T = Q$ (*see*, e.g., reference 1). In terms of T, then,

$$EV - RP = E[(Tz)'(Tz) - (TEz)'(TEz)],$$

which is just the sum of the variances of the random variables $(Tz)_i$. Thus,

$$0 \leqq EVPI \leqq \textstyle\sum_i \operatorname{var}(Tz)_i. \tag{17}$$

In the case of stochastic linear programming with quadratic recourse then, an estimate for the value of information can be written down immediately, if only we know the recourse function and the probability distribution. In particular, for this case, it is *not* necessary to solve even an approximate linear programming problem.

Example. To illustrate the notions developed in the first part of the previous section, we adopt an example from reference 9. This example is actually a modified version of an example appearing in reference 3.

Consider the case of a manufacturer who is confronted with optimal production planning with a random demand for one product. Suppose his manufacturing and market situation is described by the model min x_1 subject to $x_1 + x_2 = 100$, $x_1 = z$, $x_1 \geqq 0$, $x_2 \geqq 0$, where the random demand z is known to be between 70 and 80. Suppose he plans as if the demand would be a fixed number, e.g., $z^0 = 77$. Solution of the above program yields $x_1(z^0) = 77$, $x_2(z^0) = 23$, $\varphi(z^0) = -77$. For this value of z, also, $\pi_1^* = 0$, $\pi_2^* = 1$.

In order to estimate whether it would be worthwhile to get better information on the market before proceeding further, the manufacturer proceeds as follows. First, he estimates the probability distribution of the random demand. Suppose he estimates it as being uniform, i.e.,

$$\Pr\{z \leqq b\} = \begin{cases} 0, & \text{if } b \leqq 70, \\ 0.1b - 7, & \text{if } 70 \leqq b \leqq 80, \\ 1, & \text{if } b \geqq 80. \end{cases}$$

Second, he estimates the penalty cost and salvage value. Suppose these are taken as $\gamma = 4$ and $\delta = 0$ respectively.

According to formula (15), then,

$$Eg(z-Ax^*)=\int_{77}^{80} 0.4(b-77)\, db=1.8,$$

$$\pi^*(Ez-z^0)=1(75-77)=-2,$$

$$EVPI \leqq 1.8+2=3.8.$$

Thus, he concludes that 3.8 represents an upper bound to the amount of money that could profitably be spent in obtaining information about the market.

For so simple a problem, WS and RP can be computed exactly. We have, in fact, $WS=-75$, $RP=-78.75$, resulting in $EVPI=3.75$.

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